MATH 522 Review Notes

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April 29, 2024

Contents

0 Vector spaces and linear transformations

0.1 Any linear transformation can be represented by a matrix

Let V be a real vector space with basis $B = \{v_1, \cdots, v_n\}$. Then, $\forall v \in V$, we can write

$$v = \sum_{i=1}^{n} x_i v_i \text{ where } x_i \in \mathbb{R}.$$

We say x is the coordinates of v. Let $T : V \to V$ be a linear transformation. Define an $n \times n$ matrix $A = (a_{ij})$ by

$$Tv_j = \sum_{i=1}^{n} a_{ij} v_i \text{ where } j = 1, \cdots, n.$$

Notice that $Tv = w \iff Ax = y$ where x gives the coordinates of v and y gives the coordinates of w.

1 Metric space

1.1 Open and closed sets

Let (X, d) be a metric space and let $S \subseteq X$.

Definition 1.1 (Open set). We say S is an open set if

$$\forall p \in S \; \exists r > 0 \text{ s.t. } B(p, r) \subseteq S.$$

Definition 1.2 (Closed set). We say S is closed if its complement in X, $X \setminus S$ is open.

Proposition 1.3 (Closed set). S is closed $\iff \forall p_n \in S$ s.t. $p_n \to p \in X$, we have $p \in S$.

Definition 1.4 (Limit point). We say $p \in X$ is a limit point of S if

$$\forall r > 0 \; \exists x \in S \setminus \{p\} \text{ s.t. } x \in B(p, r).$$

Proposition 1.5 (Closed set). S is closed $\iff$ S contains all its limit points.

1.2 Completeness

Definition 1.6 (Completeness). The metric space (X, d) is complete if every Cauchy sequence (p_n) in X converges to an element $p \in X$.

1.3 Compactness

1.3.1 Open covers

Definition 1.7 (Compact set). We say $K \subseteq X$ is compact if any open cover of K can be reduced to a finite subcover.

1.3.2 Sequential compactness

Definition 1.8 (Sequential compactness). We say $K \subseteq X$ is sequentially compact if any sequence in K has a subsequence that converges to a point of K.

In metric spaces compactness and sequential compactness are equivalent.

Theorem 1.9. A set $K \subseteq (X, d)$ is compact $\iff$ it is sequentially compact.

1.4 Closed and bounded

Proposition 1.10. If $K \subseteq (X, d)$ is compact, then K is closed and bounded.

1.4.1 Heine-Borel Theorem

Theorem 1.11 (Heine-Borel Theorem). In $(\mathbb{R}^n, |x - y|)$, any closed and bounded set is compact. *Note: this is not true in a general metric space.*

2 Continuous functions on metric spaces

2.1 Continuity

2.1.1 $\epsilon - \delta$

Definition 2.1 (Continuity). $f : X \to Y$ is continuous at $a \in X$ if

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ s.t. } d(x, p) < \delta \implies d(f(x), f(p)) < \epsilon.$$

2.1.2 Sequences

Proposition 2.2 (Sequential continuity). $f : X \to Y$ is continuous at $a \in X \iff \forall$ sequence (x_n) in X,

$$x_n \to a \implies f(x_n) \to f(a).$$

2.1.3 Open sets

Proposition 2.3. $f : X \to Y$ is continuous at $a \in X \iff$ if $\mathcal{O}$ is any open set containing $f(a)$, then the preimage $f^{-1}(\mathcal{O})$ contains $B(a, \delta)$ for some $\delta > 0$.

2.2 Uniform continuity

Definition 2.4 [Uniform continuity]. $f : X \to Y$ is uniformly continuous on X if

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ s.t. } d(x_1, x_2) < \delta \implies d(f(x_1), f(x_2)) < \epsilon \; \forall x_1, x_2 \in X.$$

Proposition 2.5. Let (X, d) be compact and suppose $f : X \to Y$ is continuous. Then f is uniformly continuous on X.

Remark 2.6. A continuous function on a compact set $K \subseteq X$ is uniformly continuous on X.

2.2.1 Extreme Value Theorem

Proposition 2.7. Let (X, d) be compact and suppose $f : X \to Y$ is continuous. Then $f(X)$ is compact.

Corollary 2.8 (Extreme Value Theorem). Let (X, d) be compact and suppose $f : X \to \mathbb{R}$ is continuous. Then f attains an absolute max and an absolute min on X.

3 Differentiability

3.1 Definition of derivative

3.1.1 $\mathbb{R}^n \to \mathbb{R}$

Definition 3.1 (Differentiability). Let $f : \mathbb{R}^n \to \mathbb{R}$. We say f is differentiable at $a \in \mathbb{R}^n$ if $\exists c \in \mathbb{R}^n$ s.t. the function defined by

$$f(a + h) = f(a) + c \cdot h + r(h)$$

satisfies $\lim_{h\to 0} \frac{r(h)}{|h|} = 0$.

3.1.2 $\mathbb{R}^n \to \mathbb{R}^m$

Definition 3.2 (Differentiability). Let $F = (f_1, \cdots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$. We say F is differentiable at $a \in \mathbb{R}^n$ if $\exists C \in \mathcal{M}_{m\times n}$ s.t. the function defined by

$$F(a + h) = F(a) + C \cdot h + r(h)$$

satisfies $\lim_{h\to 0} \frac{r(h)}{|h|} = 0$.

3.1.3 $V \to W$

Definition 3.3 (Differentiability). Let $f : V \to W$ where V, W are real normed vector spaces (possibly infinitely dimensional). We say f is differentiable at $a \in V$ if $\exists$ a linear transformation $T_a : V \to W$ s.t. the function defined by

$$f(a + h) = f(a) + T_a(h) + r(h)$$

satisfies $\lim_{h\to 0} \frac{r(h)}{|h|_V} = 0_W$.

3.2 Criterion for differentiability

Theorem 3.4 Let $\mathcal{O} \subseteq \mathbb{R}^n$ be an open set and suppose $f : \mathcal{O} \to \mathbb{R}$. Suppose $f \in C^1(\mathcal{O})$. Then f is differentiable.

3.3 Important tricks

Let $f : (a, b) \to \mathbb{R}$ be C^1. Then for $x, x + y \in (a, b)$, we have the following tricks.

Trick 3.5 (Using FTC).

$$f(x + y) - f(x) = \left(\int_0^1 f'(x + ty)dt \right) y.$$

Trick 3.6 (Using MVT).

$$f(x + y) - f(x) = f'(c)y$$

for some c between $x, x + y$.

3.4 The Chain Rule

Theorem 3.7 (The Chain Rule). Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $x \in \mathbb{R}^n$. Let $G : \mathbb{R}^m \to \mathbb{R}^k$ be differentiable at $z \equiv F(x)$. Then $H = G \circ F : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable at X and

$$DH(x) = DG(F(x)) \cdot DF(x),$$

where

$$D(F(a)) = \begin{pmatrix} \nabla f_1(a) \\ \vdots \\ \nabla f_m(a) \end{pmatrix}.$$

3.5 Clairaut's Theorem

Theorem 3.8 (Clairaut's Theorem). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be C^2. Then

$$\partial_j \partial_k F(x) = \partial_k \partial_j F(x) \; \forall x.$$

3.6 Taylor's Theorem

We first introduce the multi-index notation.

3.6.1 Multi-index notation

Consider $X = (x_1, \cdots, x_n) \in \mathbb{R}^n$.

Definition 3.9 (Multi-index notation).

1. A multi-index is an n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ where $\alpha_j \in \mathbb{N}_0$.
2. Define $x^\alpha \equiv x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. In addition, define $x_j^0 \equiv 1$ even if $x_j = 0$.
3. The order of α is $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$.
4. Define $\alpha! \equiv \alpha_1!\alpha_2! \cdots \alpha_n!$. In addition, define $0! \equiv 1$.

Remark 3.10 (Polynomial). Any polynomial $p(x)$ of order $\leq m$ can be written as

$$p(x) = \sum_{|\alpha| \leq m} c_\alpha x^\alpha \text{ where } c_\alpha \text{ constant.}$$

Definition 3.11. Let

$$D = (\partial_{x_1}, \cdots, \partial_{x_n}) = (\partial_1, \cdots, \partial_n).$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ and suppose f is C^m. We define

$$D^\alpha = \partial_1^{\alpha_1} \circ \cdots \circ \partial_n^{\alpha_n}.$$

3.6.2 Multinomial Theorem

We first recall the Binomial Theorem.

Theorem 3.12 (Binomial Theorem).

$$(x_1 + x_2)^m = \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} x_1^{m-j} x_2^j.$$

Theorem 3.13 (Multinomial Theorem).

$$(x_1 + \cdots + x_n)^m = \sum_{|\alpha| = m} \frac{m!}{\alpha!} x^\alpha.$$

3.6.3 Taylor's Theorem

We first recall the single variate Taylor's Theorem.

Theorem 3.14 (Taylor's Theorem). Let $m \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ and suppose $f \in C^{m+1}$. Let $a, x \in \mathbb{R}$. We have

$$f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(m+1)}(\xi)}{(m+1)!}(x-a)^{m+1},$$

where ξ is strictly between a, x.

Theorem 3.15 (Taylor's Theorem). Let $m \in \mathbb{N}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ and suppose $f \in C^{m+1}$. Let $a, x \in \mathbb{R}^n$. We have

$$f(x) = \sum_{|\alpha| \leq m} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + \sum_{|\alpha|=m+1} \frac{D^\alpha f(\xi)}{\alpha!}(x-a)^\alpha,$$

where ξ is strictly between a, x, i.e., on the open segment joining a and x.

4 Inverse and Implicit Function Theorems

4.1 Contraction Mapping Theorem

Consider the metric space (X, d).

Definition 4.1 (Contraction). A map $\phi : X \to X$ is a contraction if $\exists 0 < c < 1$ s.t.

$$d(\phi(x) - \phi(y)) \leq cd(x, y) \; \forall x, y \in X.$$

Theorem 4.2 (Contraction Mapping Theorem). Let (X, d) be a nonempty and complete metric space. Suppose $\phi : X \to X$ is a contraction. Then $\exists$ a unique $x \in X$ s.t. $\phi(x) = x$ and we call x a fixed point.

4.2 Inverse Function Theorem

4.2.1 Between euclidean space

Theorem 4.3 (Inverse Function Theorem). Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 on $\mathbb{R}^n$. Let $a \in \mathbb{R}^n$. We have the following.

1. If $f'(a) \in \mathcal{M}_{n\times n}$ is invertible, then $\exists$ open sets $U \ni a$ and $V \ni f(a) = b$ s.t. $f : U \to V$ is a C^1-diffeomorphism, i.e., f is one-to-one, onto, and both f and f^{-1} are C^1.

2. Let $g = f^{-1} : V \to U$ then g is C^1 and

$$g'(f(x)) = [f'(x)]^{-1} \ \forall x \in U$$

4.2.2 Between normed real vector spaces

Theorem 4.4 (Inverse Function Theorem). Let V be a finitely dimensional real normed vector space. Suppose $f : V \to V$ is C^1 on V. Let $a \in V$.

Then if $f'(a) \in L(V, V)$ is invertible, then $\exists$ open sets $U_1 \ni a$ and $U_2 \ni f(a) = b$ s.t. $f : U_1 \to U_2$ is a C^1-diffeomorphism.

4.2.3 Important usages of the Inverse Function Theorem

TLDR: level sets in $\mathbb{R}^n$ can be parametrized using $n - 1$ parameters.

Example 4.5 (Surface flattening). Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Let $S = \{x \in \mathbb{R}^n : \phi(x) = 0\}$. Suppose $a \in S$, i.e., $\phi(a) = 0$ and $\nabla\phi(a) \neq 0$. Without loss of generality, assume $\phi_{x_n} \neq 0$. We want to show $\exists$ open sets $U \ni a, V \ni 0$ and a C^1 function $\psi : U \to V$ s.t.

$$\psi(a) = 0 \text{ and } \psi(S \cap U) = \{y \in V : y_n = 0\}.$$

Note: intuitively, by change of coordinates from $x = (x_1, \cdots, x_n)$ to $y = (y_1, \cdots, y_n)$, the level "surface" S in $\mathbb{R}^n_x$ is "flattened" to a "plane" in $\mathbb{R}^n_y$ where $y_n = 0$.

Proof. Define

$$\psi(x) = (x_1 - a_1, \cdots, x_{n-1} - a_{n-1}, \phi(x)).$$

Note that ψ is apparently C^1 and $\psi(a) = 0$. In addition, it is easy to verify that

$$\det \psi'(a) = \begin{vmatrix} I_{(n-1)\times(n-1)} & 0 \\ 0 & \phi_{x_n}(a) \end{vmatrix} = \phi_{x_n}(a) \neq 0.$$

Therefore, $\exists$ open sets $U \ni a, V \ni 0$ s.t. $\psi : U \to V$ is a C^1-diffeomorphism. Lastly, since $S \cap U = \{x \in U : \phi(x) = 0\}$, $\psi(S \cap U) = \{y \in V : y_n = 0\}$. $\square$

Example 4.6 (Surface parametrization). From Example 5.5, we can define $r(t)$ where $t = (t_1, \cdots, t_{n-1}) \in \mathbb{R}^{n-1}$ by $r(t) = \psi^{-1}(t_1, \cdots, t_{n-1}, 0)$ for t in a small enough $I \ni 0$. *Note: this can also be done using the Implicit Function Theorem.*

4.3 Implicit Function Theorem

Theorem 4.7 (Implicit Function Theorem). Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be C^1. In addition, we write $f(x, y)$ with $x \in \mathbb{R}^n, y \in \mathbb{R}^m$. Suppose $f(a, b) = 0$ and assume $D_x f(a, b) \equiv A_x$ is invertible. Then

1. $\exists$ open sets $U \ni (a, b)$ in $\mathbb{R}^{n+m}$ and open sets $W \ni b$ in $\mathbb{R}^m$ and C^1 function $g : W \to \mathbb{R}^n$ s.t.

   $$\{(x, y) \in U : f(x, y) = 0\} = \{(g(y), y) : y \in W\}.$$

   Note: the former is a level set of f whereas the latter is the graph of g on W. Intuitively, a level set of $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ can be locally represented by a function $g : \mathbb{R}^m \to \mathbb{R}^n$ under certain assumptions. This is not mathematically rigorous but one can think of this as: once n parameters are determined, one is only left with m "degrees of freedom" to parametrize the level set with.

2. If $A_y = D_y f(a, b)$, then $g'(b) = -A_x^{-1} A_y$.

Remark 4.8 (Special case). Let $\phi \in C^1(\mathbb{R}^3, \mathbb{R}^1)$. Let $S = \{x \in \mathbb{R}^3 : \phi(x) = 0\}$. Assume $\nabla\phi(a) \neq 0$. Without loss of generality, assume $\phi_{x_3}(a) \neq 0$. Implicit Function Theorem implies that $\exists$ open set $U \ni a = (a_1, a_2, a_3)$ in $\mathbb{R}^3$ and open set $W \ni (a_1, a_2)$ in $\mathbb{R}^2$ and a C^1 function $g : W \to \mathbb{R}$ s.t.

$$U \cap S = \{(x_1, x_2, g(x_1, x_2)) : (x_1, x_2) \in w\}.$$

4.4 Lagrange multipliers

We consider the case with one constraint.

Proposition 4.9 (Lagrange multiplier). Let $f, g \in C^1(\mathbb{R}^3, \mathbb{R})$. Let $S = \{x \in \mathbb{R}^3 : g(x) = 0\}$. Let $a \in S$ and assume $\nabla g(a) \neq 0$. If $f|_S$ has a local maximum at $a \in S$, then $\exists \lambda \in \mathbb{R}$ s.t.

$$\nabla f(a) = \lambda g(a).$$

5 Riemann integral

5.1 Integration

Let $f : \mathcal{R} \to \mathbb{R}$ bdd. Let $\mathcal{P} = \{R_1, \cdots, R_N\}$ be a partition of $\mathcal{R}$.

Definition 5.1 (Upper and lower sums). The upper sum of f associated with $\mathcal{P}$ is given by

$$U(f, \mathcal{P}) = \sum_{i=1}^{N} (\sup_{R_j} f) V(R_j).$$

The lower sum of f associated with $\mathcal{P}$ is given by

$$L(f, \mathcal{P}) = \sum_{i=1}^{N} (\inf_{R_j} f) V(R_j).$$

Definition 5.2 (Upper and lower integrals). The upper integral of f is given by

$$\overline{I}(f) = \inf_{\text{all } \mathcal{P}} U(f, \mathcal{P}).$$

The lower integral of f is given by

$$\underline{I}(f) = \sup_{\text{all } \mathcal{P}} L(f, \mathcal{P}).$$

Definition 5.3 (Riemann integrable). Let $f : \mathcal{R} \to \mathbb{R}$ bdd. We say f is Riemann integrable on $\mathcal{R}$ if

$$\underline{I}(f) = \overline{I}(f).$$

Proposition 5.4 (The criterion). Suppose $f : \mathcal{R} \to \mathbb{R}$ bdd. Then

$$f \in \text{Riem}(\mathcal{R}) \iff \forall \epsilon > 0 \, \exists \mathcal{P} \text{ s.t. } U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

Proposition 5.5. Let f be cts on $\mathcal{R}$, then $f \in \text{Riem}(\mathcal{R})$.

5.2 Jordan content

Notation 5.6 (Characteristic function). Let $S \subseteq \mathcal{R}$. Let $\chi_S : \mathcal{R} \to \mathbb{R}$ be

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \in \mathcal{R} \setminus S \end{cases}.$$

Definition 5.7 (Upper and lower content). The upper content of S is given by

$$\begin{aligned} \mathrm{cont}^+(S) = \overline{I}(\chi_S) &= \inf_{\mathcal{P}} \left\{ \sum_{j=1}^{N} (\sup_{R_j} \chi_S) V(R_j) \right\} \\ &= \inf_{\mathcal{P}} \left\{ \sum_{j=1}^{N} V(R_j) : R_j \in \mathcal{P}, R_j \cap S \neq \emptyset \right\} \\ &= \inf_{\mathcal{P}} \left\{ \sum_{j=1}^{N} V(R_j) : R_j \in \mathcal{P}, S \subseteq R_1 \cup \cdots \cup R_N \right\}. \end{aligned}$$

The lower content of S is given by

$$\begin{aligned} \mathrm{cont}_-(S) = \underline{I}(\chi_S) &= \sup_{\mathcal{P}} \left\{ \sum_{j=1}^{N} (\inf_{R_j} \chi_S) V(R_j) \right\} \\ &= \sup_{\mathcal{P}} \left\{ \sum_{j=1}^{N} V(R_j) : R_j \in \mathcal{P}, R_1 \cup \cdots \cup R_N \subseteq S \right\}. \end{aligned}$$

Definition 5.8 (Jordan content). We say S has content if

$$\mathrm{cont}^+(S) = \mathrm{cont}_-(S) \iff \chi_S \in \mathrm{Riem}(\mathcal{R}).$$

Definition 5.9 (Closure). $\overline{S}$ is the smallest closed set containing $S \iff$

$$\overline{S} = \{p \in X : \exists x_n \in S \text{ with } x_n \to p\}.$$

Definition 5.10 (Interior). $\mathring{S}$ is the largest open set contained in $S \iff$

$$\mathring{S} = \{x \in S : \exists \delta > 0 \text{ s.t. } B(x, \delta) \subseteq S\}.$$

Definition 5.11 (Boundary).

$$bS = \overline{S} \setminus \mathring{S} = \{p \in X : \forall \delta > 0, B(p, \delta) \text{ meets both } \overline{S}, S^c\}.$$

Proposition 5.12.

$$S \subseteq \mathcal{R} \text{ has content } \iff \text{cont}^+(bS) = 0 \iff \text{cont}_-(bS) = 0.$$

Definition 5.13. If $S \subseteq \mathcal{R}$ and $\text{cont}^+(S) = 0$, we say S is nil.

Proposition 5.14 (Continuous except on a nil set). Let $f : \mathcal{R} \to \mathbb{R}$ bdd. Let $S = \{x \in \mathcal{R} : f \text{ is discontinuous at } x\}$. If $\text{cont}(S) = 0$, then $f \in \text{Riem}(\mathcal{R})$.

Definition 5.15 (Integration on sets with content). Let $K \subseteq \mathcal{R}$ be closed and with content, i.e., bK is nil. Let $f : K \to \mathbb{R}$ cts. Let

$$\tilde{f}(x) = \begin{cases} f(x) & x \in K \\ 0 & x \in \mathcal{R} \setminus K \end{cases}.$$

By Proposition 1.14, define

$$\int_K f dV = \int_{\mathcal{R}} \tilde{f} dV.$$

5.3 Riemann sums

Definition 5.16 (Riemann sums). Let $f : \mathcal{R} \to \mathbb{R}$ bdd.

a) Let $\mathcal{P}$ be a partition of $\mathcal{R}$. Pick $x_j \in R_j$. The Riemann sum

$$R(f, \mathcal{P}) = \sum_{R_j \in \mathcal{P}} f(x_j) V(R_j).$$

b) Let $L \in \mathbb{R}$. We say

$$\lim_{|\mathcal{P}| \to 0} R(f, \mathcal{P}) = L$$

if $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. for any $|\mathcal{P}| < \delta$, we have $|R(f, \mathcal{P}) - L| < \epsilon$.

c) We say f is Riemann integrable (in the new sense) if $\exists L \in \mathbb{R}$ s.t.

$$\lim_{|\mathcal{P}|\to 0} R(f,\mathcal{P}) = L.$$

Proposition 5.17.

$$f \in \text{Riem}(\mathcal{R}) \text{ and } \int_{\mathcal{R}} f = L \iff f \in \text{Riem}_2(\mathcal{R}) \text{ and } \lim_{|\mathcal{P}|\to 0} R(f,\mathcal{P}) = L.$$

5.4 Fubini's Theorem

Proposition 5.18 (Modulus of continuity). Let (X,d) be a metric space. Then, $f : X \to \mathbb{R}$ is uniformly cts $\iff \exists$ monotonic function $\omega : [0,1) \to [0,\infty)$ s.t. if $\delta \searrow 0$ then $\omega(\delta) \searrow 0$ and s.t. if $d(x,y) \leq \delta < 1$, then $|f(x) - f(y)| \leq \omega(\delta)$.

Theorem 5.19 (Fubini's Theorem). Let $\Sigma \subseteq \mathbb{R}^{n-1}_x$ closed and bounded, has content, i.e., $\text{cont}(b\Sigma) = 0$. Let $g_0, g_1 : \Sigma \to \mathbb{R}$ cts. Assume $g_0 < g_1$ on Σ. Let $\Omega = \{(x,y) \in \mathbb{R}^n, x \in \Sigma, g_0(x) \leq y \leq g_1(x)\}$. Then,

a) Ω has content.

b) If $f : \Omega \to \mathbb{R}$ cts, then

$$\phi(x) = \int_{g_0(x)}^{g_1(x)} f(x,y)dy$$

is cts on Σ.

c)

$$\int_\Omega f dV_n = \int_\Sigma \phi(x) dV_{n-1} = \int_\Sigma \int_{g_0(x)}^{g_1(x)} f(x,y)dy dV_{n-1}.$$

5.5 Change of Variable Theorem

Theorem 5.20 (Change of Variable). Let $\mathcal{O}_x, \Omega_y$ be open in $\mathbb{R}^n$. Suppose $G : \mathcal{O} \to \Omega$ is a C^1 diffeomorphism. Let $f : \Omega \to \mathbb{R}$ be continuously compact supported in Ω, i.e., $f \in C_C(\Omega, \mathbb{R})$. Then

$$\int_\Omega f(y)dV(y) = \int_{\mathcal{O}} f(G(x))|\det G'(x)|dV(x).$$

6 Surfaces and surface integrals

6.1 Surfaces

Definition 6.1 (C^k $m-$dimensional surface in $\mathbb{R}^n$). Suppose $m \leq n$. A set $M \subseteq \mathbb{R}^n$ is a C^k $m-$dimensional surface in $\mathbb{R}^n$ if: given any $p \in M$ $\exists$ open set U in M with $U \ni p$, open set $\mathcal{O} \subseteq \mathbb{R}^m$ and C^k map $\phi : \mathcal{O} \to \mathbb{R}^n$ which maps bijectively to U with $\phi'(x) : \mathbb{R}^m \to \mathbb{R}^n$ injective $\forall x \in \mathcal{O}$, and $\phi^{-1} : U \to \mathcal{O}$ cts.

We call $\phi : \mathcal{O} \to U$ a coordinate chart and U a coordinate patch on M.

Definition 6.2 (Tanget spaces). Let $M \subseteq \mathbb{R}^n$ be a C^k $m-$dimensional surface. Let $\phi : \mathcal{O} \to \mathbb{R}^n$ be a chart. Say $\phi(x_0) = p$. Recall $\phi'(x_0) : \mathbb{R}^m \to \mathbb{R}^n$ injective. Define

$$T_p M = \text{Range of } \phi'(x_0) : \mathbb{R}^m \to \mathbb{R}^n.$$

6.2 Surface integrals

Definition 6.3 (Metric tensor). Let $\mathcal{O} \subseteq \mathbb{R}^m$ open with $m \leq n$. Let $\phi : \mathcal{O} \to \mathbb{R}^n$ be a C^1 chart on surface $M \subseteq \mathbb{R}^n$.

a) Define
$$G(x) = \phi'(x)^T \phi'(x) = (G_{jk}(x)))_{j,k=1}^m$$
to be the metric tensor of surface M on $U = \phi(\mathcal{O})$.

b) Define $g(x) = \det G(x)$.

Definition 6.4 (Surface integral). Suppose $f : M \to \mathbb{R}$ cts, supp $f \subseteq U = \phi(\mathcal{O})$ cpct. Define

$$\int_M f dS = \int_{\mathcal{O}} f \circ \phi(x) \sqrt{g(x)} dV(x).$$

6.3 Jordan content on surfaces

Definition 6.5 ($\text{Riem}_C(M)$).

a) Suppose $f : U \to \mathbb{R}$ bdd with supp $f \subseteq U$ cpct. Then say $f \in \text{Riem}_C(U)$ if $f \circ \phi \in \text{Riem}_C(\mathcal{O})$. If this is so, define

$$\int_U f dS = \int_{\mathcal{O}} (f \circ \phi)\sqrt{g} dx.$$

b) Let $f : M \to \mathbb{R}$ bdd with compact support. We say $f \in \text{Riem}_C(M)$ if $\exists$ a finite cover of supp f by coordinate patches $\phi_i : \mathcal{O}_i \to U_i$ and partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$ s.t. $f\rho_i \in \text{Riem}_C(U_i)$. Define

$$\int_M f dS = \sum_{i=1}^{N} \int_{U_i} f \rho_i dS.$$

Definition 6.6 ($m-$dimensional Jordan surface content). Let $\Sigma \subseteq M$ where $\overline{\Sigma}$ is cpct. We say Σ has $m-$dimensional Jordan surface content if $\chi_\Sigma \in \text{Riem}_C(M)$, in which case, define the $m-$dimensional Jordan surface content

$$A_m(\Sigma) = \int_M \chi_\Sigma dS.$$

Proposition 6.7 Let $f : M \to \mathbb{R}$ bdd with compact support on M. Let $\Sigma = \{x \in M : f \text{is discontinuous at } x\}$. If $A_m(\Sigma) = 0$, then $f \in \text{Riem}_C(M)$.

6.4 Maps between surfaces

Let M_m, N_l be C^1 surfaces in $\mathbb{R}^n$. Let $f : M \to N$. We give two equivalent definitions of C^1 maps from M to N.

Definition 6.8 (Using extensions). We say f is C^1 if $\forall p \in M$ $\exists$ open set $U \ni p$ in $\mathbb{R}^n$ s.t. $f|_{M\cap U}$ extends to a C^1 function $\tilde{f} : U \to \mathbb{R}^n$.

Definition 6.9 (Using charts). Let $\phi : \mathcal{O} \to U \subseteq M$, $\psi : \Omega \to V \subseteq N$ be C^1 charts. Define $F = \psi^{-1} \circ f \circ \phi : \mathcal{O} \to \Omega$. We say f is C^1 if $F : \mathcal{O} \to \Omega$ is C^1 for any such pair of charts.

Let f be C^1. We want to define $f'(p) : T_pM \to T_{f(p)}N$ s.t. $f'(p)$ agrees with the old definition and follows the Chain Rule. We give two definitions below.

Definition 6.10 (Using extensions). Let $\tilde{f}$ be a C^1 extension of f to an open set $U \ni p$ in $\mathbb{R}^n$. We define

$$f'(p) = \tilde{f}'(p)|_{T_pM}.$$

Definition 6.11 (Using charts). Define $h = \psi^{-1} \circ f \circ \phi$ as in Definition 2.9. Suppose $\phi(x_0) = p \in M, \psi(y_0) = f(p) \in N, y_0 = h(x_0)$. We define

$$f'(p) = \psi'(y_0) \circ h'(x_0) \circ (\phi'(x_0))^{-1}.$$

7 Multilinear forms on vector spaces

7.1 Multilinear forms

Definition 7.1 (Multilinear $k-$forms). A multilinear $k-$form on V is a function $\alpha : V^k \to \mathbb{R}$ that is linear in each argument when the others are held fixed.

Write $\mathcal{T}^k(V)$ for the vector space of all multilinear $k-$forms on V.

Definition 7.2 (Tensor product). If $\alpha \in \mathcal{T}^p(V)$ and $\beta \in \mathcal{T}^q(V)$, we define the tensor product $\alpha \otimes \beta \in \mathcal{T}^{p+q}(V)$ by

$$\alpha \otimes \beta(v, w) = \alpha(v)\beta(w) \text{ for } v \in V^p, w \in V^q.$$

Definition 7.3 (Pullbacks). Let $A : V \to W$ be a linear transformation and suppose $\beta \in \mathcal{T}^k(W)$. THen, we define $A^*\beta \in \mathcal{T}^k(V)$ by

$$A^*\beta(v) = \beta(Av).$$

Definition 7.4 (Basis and dual basis). Let $B = \{v_1, \cdots, v_n\}$ be a basis of V. Let $\omega_i \in \mathcal{T}^1(V)$ be the linear functional satisfying

$$\omega_i(v_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}.$$

Then, $B' = \{\omega_1, \cdots, \omega_n\}$ is a basis of $\mathcal{T}^1(V)$.

7.2 Alternating mutilinear forms

Definition 7.5 (Alternating multilinear $k-$forms). A multilinear $k-$form α is alternating if the sign of α is reversed whenever two arguments are transposed.

We denote by $\Lambda^k(V) \subseteq \mathcal{T}^k(V)$ the subspace of alternating multilinear $k-$forms.

Definition 7.6 (Wedge product). For $\alpha \in \Lambda^p(V)$ and $\beta \in \Lambda^q(V)$, we define the wedge product $\alpha \wedge \beta \in \Lambda^{p+q}(V)$ by

$$\alpha \wedge \beta = \mathrm{Alt}(\alpha \otimes \beta).$$

Proposition 7.7 (Basis of $\Lambda^k(V)$)

a) Let $\mathcal{I}_k$ denote the set of all k-tuples $I = (i_1, \cdots, i_k)$, where each $i_p \in \{1, \cdots, n\}$.

b) Suppose $\dim V = n$ and let $B' = \{\omega_i : i = 1, \cdots, n\}$ be a basis of $\mathcal{T}^1(V)$. For $I = (i_1, \cdots, i_k) \in \mathcal{I}_k$, we set

$$\omega_{I,\otimes} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_k} \in \mathcal{T}^k(V).$$

c) Let $\omega_I = \mathrm{Alt}\omega_{I,\otimes} = \omega_{i_1} \wedge \cdots \wedge \omega_{i_k} \in \Lambda^k(V)$.

d) If $k \leq n$, let $\mathcal{I}_{k,\nearrow} \subseteq \mathcal{I}_k$ denote the subset of k-tuples I satisfying $i_1 < \cdots < i_k$.

Suppose $\dim V = n$. Let $k \leq n$. A basis of $\Lambda^k(V)$ is given by

$$\{\omega_I : I \in \mathcal{I}_{k,\nearrow}\}.$$

7.3 Determinant

Definition 7.8 (Determinant). Let $\{e_i : i = 1, \cdots, n\}$ be the standard basis of $\mathbb{R}^n$. We denote by det the unique element of $\Lambda^n(R^n)$ such that $\det(e_1, \cdots, e_n) = 1$. Let $B' = \{\omega_i : i = 1, \cdots, n\}$ be the dual basis of the standard basis of $\mathbb{R}^n$. Then

$$\det = \omega_1 \wedge \cdots \wedge \omega_n.$$

Proposition 7.9 (Classical formula for determinant). Let $a_j \in \mathbb{R}^n$. Write $a_j = (a_{1j}, \cdots, a_{nj})$. Then,

$$\det(a_1, \cdots, a_n) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

Definition 7.10 (Determinant of a linear transformation). Suppose $\dim V = n$, $B = \{v_1, \cdots, v_n\}$ is a basis of V, and $B' = \{\omega_1, \cdots, \omega_n\}$ is the dual basis of $\Lambda^1(V)$. Suppose $T : V \to V$ is a linear transformation. Then

$$T^*(\omega_1 \wedge \cdots \wedge \omega_n) = (\det T)\omega_1 \wedge \cdots \wedge \omega_n.$$

7.4 Orientation of a vector space

We define equivalence relation on $\Lambda^k(V) \setminus \{0\}$ by declaring $\alpha \sim \beta$ when α is a positive scalar multiple of β. If $\gamma \in \Lambda^k(V) \setminus \{0\}$ is a given fixed element, we write

$$\Lambda^k(V) \setminus \{0\} = \Lambda^k_+(V) \cup \Lambda^k_-(V),$$

where $\Lambda^k_+(V)$ consists of all β such that $\beta \sim \gamma$ and $\Lambda^k_-(V)$ consists of all β s.t. $\beta \sim -\gamma$.

Definition 7.11 (By choice of ω). Each of the equivalence classes $\Lambda^k_+(V), \Lambda^k_-(V)$ is said to be an orientation of V. Any element $\omega \in \Lambda^k_+(V)$ is said to determine the positive orientation.

Proposition 7.12 (By choice of ordered basis). Let $B = \{v_1, \cdots, v_k\}$ be an ordered basis of V. We say B fixes the same orientation as ω if

$$\omega(v_1, \cdots, v_k) > 0.$$

8 Differential forms

8.1 Forms

Definition 8.1 (Alternating $k-$form on a surface). An alternating $k-$form on surface M is a function ω s.t. for $p \in M$ we have

$$\omega(p) \in \Lambda^k(T_pM),$$

i.e.,

$$\omega : M \to \bigcup_{p \in M} \Lambda^k(T_pM).$$

Definition 8.2 (Differential forms on a surface). If ω is smooth, then we say ω is a differential $k-$form on M. We write $\omega \in \Lambda^k(M)$. If $k = 0$, define $\Lambda^0(M) = C^\infty(M, \mathbb{R})$.

Proposition 8.3 (Properties of differential forms).

a) If $\omega_1, \omega_2 \in \Lambda^k(M)$, then $\omega_1 + \omega_2 \in \Lambda^k(M)$ given by

$$(\omega_1 + \omega_2)(p) = \omega_1(p) + \omega_2(p).$$

b) If $c \in \mathbb{R}$, then $c\omega_1 \in \Lambda^k(M)$ given by

$$(c\omega_1)(p) = c\omega_1(p).$$

c) Let $\omega \in \Lambda^p(M), \theta \in \Lambda^q(M)$. Define $\omega \wedge \theta \in \Lambda^{p+q}(M)$ given by

$$(\omega \wedge \theta)(p) = \omega(p) \wedge \theta(p).$$

If $\omega \in \Lambda^0(M)$, define $\omega \wedge \theta = \omega\theta \in \Lambda^q(M)$ where

$$(\omega\theta)(p) = \omega(p)\theta(p).$$

Definition 8.4 (Pullbacks). Let M, N be smooth surfaces in $\mathbb{R}^n$. Suppose $f : M \to N$ is $C^\infty, p \in M, f(p) \in N, f'(p) : T_pM \to T_{f(p)}N$. Let $\omega \in \Lambda^k(N)$. Define $f^*\omega \in \Lambda^k(M)$ by

$$(f^*\omega)(p) = \begin{cases} f'(p)^*\omega(f(p)) & k \geq 1 \\ \omega \circ f & k = 0 \end{cases}$$

Proposition 8.5 (Properties of pullbacks). Let $f : M \to N, h : P \to M$, then $f \circ h : P \to N$.

a) $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$.

b) $f^*(\omega \wedge \theta) = f^*\omega \wedge f^*\theta$.

c) $(f \circ h)^*\omega = h^* f^*\omega$ where $\omega \in \Lambda^k(M)$.

8.2 Differentials

Definition 8.6 (Differentials). Let $\omega \subseteq \mathbb{R}^n$ open and $x_0 \in \mathcal{O}$. Suppose $f \in C^\infty(\mathcal{O}, \mathbb{R}) = \Lambda^0(\mathcal{O})$. Since

$$f : \mathcal{O} \to \mathbb{R},$$

we have

$$f'(x_0) : T_{x_0}\mathcal{O} \to T_{f(x_0)}\mathbb{R} = \mathbb{R}.$$

Hence, $f'(x_0) \in \Lambda^1(T_{x_0}\mathcal{O})\ \forall x_0 \in \mathcal{O}$. We write

$$f'(x_0) = df(x_0), f' = df.$$

Notice the differential $1-$form on $\mathcal{O}$ $df \in \Lambda^1(\mathcal{O})$. Call it the differential of f.

Proposition 8.7. Let $x_0 \in \mathcal{O}$, $v \in T_{x_0}\mathcal{O} = \mathbb{R}^n$,

$$df(x_0)v = f'(x_0)v = \nabla f(x_0) \cdot v.$$

In particular, $dx_i \in \Lambda^1(\mathcal{O})$. We saw

$$dx_i(x_0)e_j = \nabla x_i(x_0) \cdot e_j = \delta_{ij}.$$

Hence, $\{dx_1(x_0), \cdots, dx_n(x_0)\}$ is a basis of $\Lambda^1(T_{x_0}\mathcal{O})$ dual to $\{e_1, \cdots, e_n\}$. So,

$$df(x_0) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x_0) dx_j(x_0).$$

Definition 8.8 (Smooth forms on $\mathcal{O}$). Let $\omega \in \Lambda^k(\mathcal{O})$, then $\omega(x_0) \in \Lambda^k(T_{x_0}\mathcal{O})$. We can write

$$\omega = \sum_{I\nearrow} a_I dx_I = \sum_{I\nearrow} a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Lambda^k(\mathcal{O}).$$

We say ω is smooth and write $\omega \in \Lambda^k(\mathcal{O})$ when a_I are smooth (C^∞).

Definition 8.9 (Forms on surfaces). Let $f \in \Lambda^0(M) = C^\infty(M, \mathbb{R})$. Define $df \in \Lambda^1(M)$ s.t. for $p \in M$,

$$df(p) = f'(p) \in \Lambda^1(T_pM).$$

Definition 8.10 (Smooth forms on surfaces). We say ω is smooth, $\omega \in \Lambda^k(M)$ when a_I are smooth for

$$\omega = \sum_{I\nearrow} a_I dx_I = \sum_{I\nearrow} a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Lambda^k(M).$$

Proposition 8.11. Let M, N be smooth surfaces in $\mathbb{R}^n$. Let $F : M \to N, F \in C^\infty(M, N), h : N \to \mathbb{R}, h \in C^\infty(N, \mathbb{R})$. Hence, $h \circ F : M \to \mathbb{R}$. For $dh \in \Lambda^1(N)$, we have

$$F^*(dh) = d(F^*h).$$

Remark 8.12 (Dual role of x_i). Proposition 4.11 implies that for coordinate chart $\phi : \mathcal{O} \to U$, where $\phi(x_0) = p$, we have

$$dx_i(x_0) = d(\phi^* x_i)(x_0) = \phi^* dx_i(x_0).$$

Here, the first occurence of $x_i : \mathcal{O} \to \mathbb{R}$ is the coordinate function on $\mathcal{O}$; the last occurence of $x_i : M \to \mathbb{R}$ is the coordinate function on M.

Proposition 8.13. Let $\phi : \mathcal{O}_x \to U \subseteq M$ and $\psi : \Omega_y \to U$ be smooth charts, and define $F : \mathcal{O} \to \Omega$ by $F = \psi^{-1} \circ \phi$. Let $\omega = a dy_1 \wedge \cdots dy_m \in A^m(U)$.

a) $\psi^*\omega = (a \circ \psi) dy_1 \wedge \cdots \wedge dy_m$.

b) $\phi^*\omega = (a \circ \phi) \det F' dx_1 \wedge \cdots \wedge dx_m$.

c) $F^*\psi^*\omega = \phi^*\omega$.

d)
$$\omega = (\phi^{-1})^*\phi^*\omega = a \det(F' \circ \phi^{-1}) dx_1 \wedge \cdots \wedge dx_m$$
and
$$\omega = (\psi^{-1})^*\psi^*\omega = a dy_1 \wedge \cdots \wedge dy_m.$$

8.3 Orientation of a surface

Definition 8.14 (Orientation of M_m). Let M_m be smooth. We say M_m is orientable if $\exists$ a nowhere vanishing element $\omega \in \Lambda^m(M_m)$, i.e., $\omega(p) \neq 0\ \forall p \in M$.

Definition 8.15 (Local orientation using charts). Not all surfaces are orientable, e.g., Mobius strip. But we can always orient a coordinate patch $U \subseteq M_m$. Take $\mathcal{O}$ to be oriented by $dx_1 \wedge \cdots \wedge dx_m$ where $x_j \in C^\infty(\mathcal{O}, \mathbb{R})$. Then, we can take

$$(\phi^{-1})^* dx_1 \wedge \cdots \wedge dx_m = dx_1 \wedge \cdots \wedge dx_m$$

to be the orientation of U where $x_j \in C^\infty(U, \mathbb{R})$.

8.4 Integration of forms

Definition 8.16 (Integration of forms on $\mathcal{O}$). Take $M_m = \mathcal{O}_x \subseteq \mathbb{R}^m$ open. Then, $dx_1 \wedge \cdots \wedge dx_m$ where $x_j \in C^\infty(\mathcal{O}, \mathbb{R})$ orients $\mathcal{O}$. Let $\omega \in \Lambda^m(\mathcal{O})$ have compact support in $\mathcal{O}$. Then, we can write $\omega = a dx_1 \wedge \cdots \wedge dx_m$ where $a \in C_C^\infty(\mathcal{O}, \mathbb{R})$. We define

$$\int_{M_m = \mathcal{O}} \omega = \int_{\mathcal{O}} a(x) dV(x).$$

Definition 8.17 (Integration of forms on M_m with compact support on U). Let M_m be an oriented smooth surface. Choose $\psi : \Omega_y \to U$ s.t. $dy_1 \wedge \cdots \wedge dy_m$ gives the prescribed orientation. We can write $\omega = a dy_1 \wedge \cdots \wedge dy_m$ where $y_j \in C^\infty(U, \mathbb{R})$. We define

$$\begin{aligned}
\int_M \omega &= \int_{\Omega_y} \psi^* \omega \\
&= \int_{\Omega_y} (a \circ \psi) dy_1 \wedge \cdots \wedge dy_m \text{ where } y_j \in C^\infty(\mathcal{O}, \mathbb{R}) \\
&= \int_{\Omega_y} a(\psi(y)) dV(y).
\end{aligned}$$

Definition 8.18 (Integration of forms on M_m with compact support on M_m). Let $\omega \in \Lambda_C^m(M_m)$. Choose charts $\phi_i : \mathcal{O}_i \to U_i$ s.t.

a) ϕ_i gives the prescribed orientation on M.

b) $\operatorname{supp} \omega \subseteq \bigcup_{i=1}^k U_i$.

Next, choose a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$ on $\operatorname{supp} \omega$. We define

$$\int_M \omega = \sum_{i=1}^k \int_M \rho_i \omega.$$

9 Generalized Stokes Theorem

9.1 Generalized Stokes Theorem

Theorem 9.1 (Generalized Stokes Theorem). Let M be an oriented $m-$dimensional surface with boundary. Let $i : \partial M \to M$ be the inclusion map. Let $\omega \in \Lambda_c^{m-1}(M)$. Give ∂M the induced orientation. Then,

$$\int_{M_m} d\omega = \int_{(\partial M)_{m-1}} i^* \omega.$$

Theorem 9.2 (Green's Theorem). Let Ω be a bounded, connected open subset of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$ oriented positively. Let $f, g \in C^\infty(\mathbb{R}^2, \mathbb{R})$. Then,

$$\int_\Omega (g_x - f_y) dx dy = \int_{\partial\Omega} f dx + g dy.$$

Remark 9.3. Let $\omega = f dx + g dy \in \Lambda^1(\Omega)$. Then

$$d\omega = (g_y - f_x) dx \wedge dy \in \Lambda^2(\Omega).$$

Theorem 9.4 (Stokes Theorem). Let S be a smooth compact oriented $2-$dimensional surface with boundary in $\mathbb{R}^3$. Let $F = (f_1, f_2, f_3) \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$. Give ∂S the induced orientation, and let n be the unit normal vector to S determined by the given orientation of S. Then,

$$\int_S (\mathrm{curl} F \cdot n) dS = \int_{\partial S} f_1 dx_1 + f_2 dx_2 + f_3 dx_3.$$

Remark 9.5. Let $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 \in \Lambda^1(\mathbb{R}^3)$. Then

$$d\omega = g_1 dx_2 \wedge dx_3 + g_2 dx_3 \wedge dx_1 + g_3 dx_1 \wedge dx_2,$$

where $(g_1, g_2, g_3) = \text{curl}(f_1, f_2, f_3)$.

Theorem 9.6 (Divergence Theorem). Let W be a bounded connected open set in $\mathbb{R}^3$ with smooth boundary ∂W and suppose $F = (f_1, f_2, f_3) \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$. Then,

$$\int_W \text{div} F dxdydz = \int_{\partial W} F \cdot n dS.$$

Remark 9.7. Let $\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2 \in \Lambda^2(\mathbb{R}^3)$. Then

$$d\omega = \text{div} F dx_1 \wedge dx_2 \wedge dx_3.$$

9.2 Closed and exact forms

Definition 9.8 (Closed and exact forms). A differential $k-$form ω on M is closed if $d\omega = 0$ and exact if $\omega = d\theta$ for some $\theta \in \Lambda^{m-1}(M)$.

Proposition 9.9. Every exact form is closed.

Proposition 9.10. Let M be an $m-$dimensional simply connected smooth surface and $\omega \in \Lambda^1(M)$. If ω is closed, then ω is also exact.

Proposition 9.11. Let M and N be compact oriented smooth surfaces of dimension m, and suppose $M = \partial W$ where W is a compact oriented smooth surface of dimension $m + 1$. Suppose $f : M \to N$ is a smooth map which extends smoothly to all of W. Then for every $\omega \in \Lambda^m(N)$, we have

$$\int_M f^* \omega = 0.$$

9.3 Brouwer Fixed Point Theorem

Definition 9.12. Let W be a smooth surface with boundary ∂W. A retraction of W onto its boundary is a map $\phi : W \to \partial W$, no necessarily smooth, such that

$$\phi(p) = p\, \forall p \in \partial W.$$

Theorem 9.13 (No Retraction Theorem). Let W be a compact smooth oriented $(m+1)-$dimensional surface with nonempty boundary ∂W. There is no smooth retraction.

Theorem 9.14 (Brouwer Fixed Point Theorem). Let $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$. Suppose $F : B \to B$ smooth. Then $\exists x \in B$ s.t. $F(x) = x$.

Definition 9.15 (Volume form). Let M be an oriented smooth $m-$dimensional surface and suppose $\phi : \mathcal{O}_x \to U \subset M$ is any orientation-preserving chart on M. We define ω_M on M by setting

$$\omega_M|_U = \sqrt{g \circ \phi^{-1}} dx_1 \wedge \cdots \wedge dx_m,$$

for any such chart, where $g = \det G, G(x) = \phi'(x)^t \phi'(x)$.

Remark 9.16. The volume form has property

$$\int_M \omega_M = \int_M dS = \text{vol}(M).$$

10 ODE Theory

We study the general $n \times n$ first-order initial value problem (IVP)

$$\frac{dy}{dt} = F(t, y),\ y(t_0) = y_0. \text{ (IVP)}$$

Theorem 10.1 (Local existence). Consider the IVP. Let $y_0 \in \Omega$, an open subset of $\mathbb{R}^n$. Let $I \subset \mathbb{R}$ be an open interval containing t_0.

1. Suppose $F : I \times \Omega \to \mathbb{R}^n$ is continuous.

2. Suppose $\exists L > 0$ s.t.

$$|F(t, y_1) - F(t, y_2)| \leq L|y_1 - y_2| \; \forall t \in I, y_j \in \Omega.$$

Then IVP has a C^1 solution on some open interval containing t_0.

Theorem 10.2 (Uniqueness). Consider the IVP. Let $I \subset \mathbb{R}$ be an open interval.

1. Suppose $F : I \times \Omega \to \mathbb{R}^n$ is continuous.
2. Suppose $\exists L > 0$ s.t.

$$|F(t, y_1) - F(t, y_2)| \leq L|y_1 - y_2| \; \forall t \in I, y_j \in \Omega.$$

Let $I' \subset I$ be an open subinterval containing t_0 on which two solutions y and z are given. Then $y = z$ on I'.

Proposition 10.3 (Uniform local existence). Consider the IVP.

1. Suppose for each compact $K \subseteq \Omega$, there exists $M_K < \infty$ s.t.

$$|F(t, x)| \leq M_K \; \forall x \in K, t \in I.$$

2. Suppose for each such K, $\exists L_K < \infty$ s.t.

$$|F(t, x) - F(t, y)| \leq L_K|x - y| \; \forall x, y \in K, t \in I.$$

Let $K \subset \Omega$ compact. Then there exists $T > 0$ s.t. for each $t_0 \in I$ and $y_0 \in K$, a unique solution to IVP exists on $[t_0 - T, t_0 + T]$. We call T a uniform time of existence for $I \times K$.

Remark 10.4. If $F \in C^1(\mathbb{R} \times \mathbb{R}^n)$, then F satisfies uniform local existennce when I is any *bounded* open interval and Ω is any *bounded, convex* open set in $\mathbb{R}^n$.

Proposition 10.5 (Criterion for global existence). Consider the IVP where F satisfies uniform local existence.

Suppose that if $J \subset I$ is any bounded open subinterval containing t_0 on which a C^1 solution y exists, there exists a compact set $K \subset \Omega$ s.t. $y(t) \in K\ \forall t \in J$. Then y extends uniquely to a C^1 solution on I.

Proposition 10.6 (Linear energy estimate). Consider a C^1 solution to the IVP

$$\frac{dy}{dy} = A(t)y + B(t),\ y(0) = y_0$$

on an interval $I \ni 0$ where $A \in C(I, M(n, \mathbb{R}))$ and $B \in C(I, \mathbb{R}^n)$. If $||A(t)|| \leq K\ \forall t \in I$, then $y(t)$ satisfies $\forall t \in I, t \geq 0$:

$$|y(t)|^2 \leq e^{(2K+1)t}|y_0|^2 + \int_0^t e^{(2K+1)(t-s)}|B(s)|^2 ds.$$

The same formula holds for $t \in I, t \leq 0$, but with $B(s)$ replaced by $B(-s)$ and t replaced by $|t|$ on the right.

[Uniqueness] Consequently, if y_1 and y_2 are C^1 solutions on I, we must have $y_1 = y_2$.

11 Compactness in function spaces

Remark 11.1. In *any* finite dimensional normed vector space, a set K is compact $\iff$ K is closed and bounded (Heine-Borel). In *any* metric spacel K compact $\implies$ K closed and bounded. However, in most function spaces, the converse of the last statement fails.

Example 11.2. Consider the metric space $C([0,1], \mathbb{R})$ with the metric associated with the sup norm, i.e., $d(f,g) = \sup_{[0,1]} |f(x) - g(x)|$. The set

$$\{x^n : n = 1, 2, \cdots\} \subset B(0,1) \subset C([0,1], \mathbb{R}).$$

Observe that $\overline{\{x^n\}}$ is closed (by construction), bounded (by the unit ball), but not compact in $C([0,1], \mathbb{R})$.

For the sake of contradiction, suppose compactness. Then, notice that any subsequence of (x^n) that converges in the above metric, must converge *uniformly* to a continuous function. But we know $x^n \to f$ point-wise and f is not continuous.

Definition 11.3 (Equicontinuity). Let (X, d) be a compact metric space. Let $\mathcal{F}$ be a family of functions $f : X \to \mathbb{R}$. We say $\mathcal{F}$ is equicontinuous if given any $\epsilon > 0$ $\exists \delta = \delta(\epsilon) > 0$ s.t. if $d(p, q) < \delta$ then $|f(p) - f(q)| < \epsilon$ $\forall f \in \mathcal{F}$.

Definition 11.4 (Density). We say A is dense in (X, d) if $\forall \epsilon > 0$ and $p \in X$, $\exists a \in A$ s.t. $d(a, p) < \epsilon$.

Proposition 11.5. (X, d) is a compact metric space $\implies$ X has a countable dense subset.

Theorem 11.6 (Arzela-Ascoli Theorem). Let (X, d) be a compact metric space. Consider $C(X, \mathbb{R})$ with its usual sup norm, i.e.,

$$|f| = \sup_{x \in X} |f(x)|.$$

Let a family of functions $K \subset C(X, \mathbb{R})$ be closed, bounded, and equicontinuous. Then K is compact.

12 Density and approximation in function spaces

12.1 Approximate identity

Proposition 12.1 (Differentiation under the integral sign). Let $\Omega \subset \mathbb{R}^2$ open. Let $R = \{(x, t) : a \leq x \leq b, c \leq t \leq d\} \subset \Omega$. Let $f \in C^1(\Omega, \mathbb{R})$. For $x \in (a, b)$ let

$$\phi(x) = \int_c^d f(x, t) dt.$$

Then,

$$\phi'(x) = \int_c^d f_x(x, t) dt$$

and ϕ is C^1 on (a, b).

Definition 12.2 (Convolutions). Let $f \in C(\mathbb{R}^n, \mathbb{R})$ and $g \in C_c(\mathbb{R}^n, \mathbb{R})$. Define

$$(f * g)(x) \equiv \int_{\mathbb{R}^n} f(x - y) g(y) dy = \int_{\mathbb{R}^n} f(y) g(x - y) dy.$$

Proposition 12.3.

a) Let $f \in C(\mathbb{R}^n, \mathbb{R}), g \in C_c^k(\mathbb{R}^n, \mathbb{R})$. Then $f * g \in C^k(\mathbb{R}^n, \mathbb{R})$ and if $|\alpha| \leq k$,

$$\partial^\alpha (f * g) = f * (\partial^\alpha g).$$

b) Let $f \in C^k(\mathbb{R}^n, \mathbb{R}), g \in C_c(\mathbb{R}^n, \mathbb{R})$. Then $f * g \in C^k(\mathbb{R}^n, \mathbb{R})$ and if $|\alpha| \leq k$,

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g.$$

Definition 12.4 (Approximate identities). Fix $g \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ s.t. $g \geq 0$, supp$g \subset \overline{B(0,1)}$, and $\int_{\mathbb{R}^n} g(x)dx = 1$. Define

$$g_k(x) \equiv k^n g(kx), \ k = 1, 2, \cdots$$

So, $g_k \geq 0$, supp$g_k \subset \overline{B(0, \frac{1}{k})}$, and $\int_{\mathbb{R}^n} g_k(x)dx = 1 \ \forall k$. We call (g_k) an approximate identity.

Proposition 12.5. Let $m \geq 0$. Let $f \in C^m(\mathbb{R}^n, \mathbb{R})$. Set

$$f_k(x) \equiv (f * g_k)(x) \in C^\infty.$$

For any compact $K \subset \mathbb{R}^n$ and $|\alpha| \leq m$, we have $\partial^\alpha f_k \to \partial^\alpha f$ uniformly on K. In particular, $f_k \to f$ as $f \to \infty$.

12.2 Frechet (metric) topology

Definition 12.6. Let $\Omega \subset \mathbb{R}^n$ open. Write $\Omega = \bigcup_{j=1}^\infty K_j$ as the union of an increasing sequence of compact subsets. For example,

$$K_j = \{x \in \Omega : \text{dist}(x, b\Omega) \geq \frac{1}{j}\} \cap \overline{B(0,j)}.$$

For each j define a seminorm on $C^k(\Omega, \mathbb{R})$ by

$$\rho_j(f) \equiv \sup_{x \in K_j, |\alpha| \leq k} |\partial^\alpha f(x)|.$$

Finally, for $f, g \in C^k(\Omega, \mathbb{R})$, define

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(f-g)}{1+\rho_j(f-g)}.$$

Remark 12.7.

a) d is a metric.

b) Let (f_n) be a sequence in $C^k(\Omega, \mathbb{R})$ and $f \in C^k(\Omega, \mathbb{R})$. Then $f_n \to f$ in the metric space $C^k(\Omega, \mathbb{R})$ if and only if given any compact set $K \subset \Omega$ and multi-index α with $|\alpha| \leq k$, the sequence $\partial^\alpha f_n \to \partial^\alpha f$ uniformly on K.

c) $C^k(\Omega, \mathbb{R})$ with the metric d is a complete metric space.

12.3 Stone-Weierstrass Theorem

Theorem 12.8 (Weierstrass Approximation Theorem). Let $f \in C([a,b], \mathbb{R})$. Then, $\exists$ polynomials p_n s.t. $p_n \to f$ uniformly on $[a,b]$ as $n \to \infty$, i.e., the set of all polynomials on $[a,b]$ is dense in $C([a,b], \mathbb{R})$.

Definition 12.9 (Algebra). An algebra of real-valued functions on a set X is a set of functions that is closed under (1) addition, (2) multiplication, and (3) scalar multiplication by $\mathbb{R}$.

Theorem 12.10 (Stone-Weierstrass Theorem). Let X be a compact metric space. Let $\mathcal{A} \subset C(X, \mathbb{R})$ be a sub-algebra. Suppose $1 \in \mathcal{A}$ and $\mathcal{A}$ separates points, i.e., if $p, q \in X, p \neq q$, then $\exists h_{pq} \in \mathcal{A}$ s.t. $h_{pq}(p) \neq h_{pq}(q)$. Then, the closure of $\mathcal{A}$ in the sup norm, $\overline{\mathcal{A}} = C(X, \mathbb{R})$.

Definition 12.11 (Self-adjoint). An algebra of function $f : X \to \mathbb{C}$ where X is a compact metric space is said to be self-adjoint if $f \in \mathcal{A} \implies \overline{f} \in \mathcal{A}$.

Theorem 12.12 (Stone-Weierstrass Theorem (complex version)). Let (X, d) be a compact metric space. Let $\mathcal{A} \subset C(X, \mathbb{C})$ be a self-adjoint sub-algebra. Suppose $1 \in \mathcal{A}$ and $\mathcal{A}$ separates points in X, then $\overline{\mathcal{A}} = C(X, \mathbb{C})$.

Definition 12.13 (Trignometric polynomials). Define the set of all trignometric polynomials to be the set $\{\sum_{|k|\leq N} a_k e^{ik\theta}, N = 0, 1, 2, \cdots, a_k \in \mathbb{C}\}$.

Example 12.14 (Fourier series). Let the set of periodic functions

$$C_p([0, 2\pi,], \mathbb{C}) = \{f \in C([0, 2\pi], \mathbb{C}), f(0) = f(2\pi)\}.$$

Then the set of all trignometric polynomials is dense in $C_p([0, 2\pi], \mathbb{C})$.

13 Lebesgue measure and integration

13.1 $\sigma-$algebra

Definition 13.1 ($\sigma-$algebra). Let X be a nonempty set. We say $\mathcal{A} \subset \mathcal{P}(X)$ (the power set) is $\sigma-$alebra on X if

1. if $E_1, E_2, \cdots \in \mathcal{A}$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$, and
2. if $E \in \mathcal{A}$, then $E^c \equiv X \setminus E \in \mathcal{A}$.

Remark 13.2 If $E_1, E_2, \cdots \in \mathcal{A}$, then $\bigcap_{j=1}^{\infty} E_j \in A$. (Proof using De Morgan's laws.)

Corollary 13.3. If $\mathcal{E} \subset \mathcal{P}(X)$. Then there is a unique smallest $\sigma-$algebra that contains $\mathcal{E}$, $\sigma(\mathcal{E})$. Call it the $\sigma-$algebra generated by $\mathcal{E}$, where

$$\sigma(\mathcal{E}) = \bigcap\{\sigma - \text{algebra that contain } \mathcal{E}\}.$$

Definition 13.4 (Borel $\sigma-$algebra)

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\{\text{open sets in } \mathbb{R}^n\})$$

13.2 Measure

Let X be a nonempty set. Let $\mathcal{M}$ be a $\sigma-$algebra on X.

Definition 13.5 (Measure). A measure μ on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ s.t.

1. $\mu(\emptyset) = 0$, and
2. [Countable additivity] if $E_j \in \mathcal{M}, j = 1, 2, \cdots$ disjoint, then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j).$$

We call $(X, \mathcal{M}, \mu)$ a measure space.

Proposition 13.6 (Properties of measures).

1. Let $E, F \in \mathcal{M}$. Then $E \subset F \implies \mu(E) \leq \mu(F)$.
2. [Subadditivity] Let $E_1, E_2, \cdots \in \mathcal{M}$ not necessarily disjoint, then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j).$$

3. [Continuity from below] Let $E_1 \subset E_2 \subset \cdots$ where $E_j \in \mathcal{M}$. Then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

13.3 Lebesgue measure

Definition 13.7 (Outer measure). An outer measure on set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ s.t.

1. $\mu^*(\emptyset) = 0$.
2. $A \subset B \implies \mu^*(A) \leq \mu^*(B)$.
3. $A_j \in \mathcal{P}(X) \implies$

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j).$$

Definition 13.8 (Lebesgue outer measure on $\mathbb{R}^n$).

Let $\mathcal{E} = \{\text{bounded open intervals in } \mathbb{R}^n\}$. An open interval $I \in \mathcal{E}$ has the form

$$I = \{x \in \mathbb{R}^n : a_i < x_i < b_i, a_i, b_i \in \mathbb{R}^n\}.$$

Let $\lambda : \mathcal{E} \to [0, \infty]$ be defined by the usual volume, i.e.,

$$\lambda(I) = \prod_{j=1}^{n}(b_j - a_j).$$

If $S \subset \mathbb{R}^n$, we define the Lebesgue outer measure

$$m^*(S) \equiv \inf \left\{ \sum_{j=1}^{\infty} \lambda(I_j) : S \subset \bigcup_{j=1}^{\infty} I_j, I_j \in \mathcal{E} \right\}.$$

Theorem 13.9. The restriction of m^* to $\mathcal{B}(\mathbb{R}^n)$ is a measure on $\mathcal{B}(\mathbb{R}^n)$. So, $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), m^*)$ is a measure space.

Definition 13.10 (Lebesgue measureable sets). Define the set of Lebesgue measurable sets

$$\mathcal{L}^n = \{E \cup F : E \in \mathcal{B}(\mathbb{R}^n), F \subset N \text{ for some } N \in \mathcal{B}(\mathbb{R}^n) \text{ s.t. } m^*(N) = 0.\}$$

Theorem 13.11. The Lebesgue outer measure restricted to the set of Lebesgue measurable sets is a measure, i.e.,

$$m^*|_{\mathcal{L}_n} \equiv m$$

is the Lebesgue measure on $\mathbb{R}^n$. So, $(\mathbb{R}^n, \mathcal{L}^n, m)$ is a measure space.

13.4 Lebesgue integration

Definition 13.12 (Lebesgue measurable functions). Let $f : (\mathbb{R}^n, \mathcal{L}^n) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We say f is Lebesgue measurable if $f^{-1}(B) \in \mathcal{L}^n\ \forall B \in \mathcal{B}(\mathbb{R})$.

Remark 13.13.

1. It is enough to check $f^{-1}((a,b)) \in \mathcal{L}^n \ \forall (a,b)$.
2. f, g measurable $\implies f+g, fg$ measurable.
3. Limit of a sequence of measurable functions is measurable. Consider $f_j, j = 1, 2, \cdots$. Then $\sup_j f_j, \inf_j f_j, \limsup_{j\to\infty} f_j, \liminf_{j\to\infty} f_j$ are measurable.

Example 13.14. Let $A \in \mathcal{L}^n$. The characteristic function

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

is measurable. To see this, notice that $\chi_A^{-1}((a,b)) = A$ or A^c or $\mathbb{R}^n$ or $\emptyset$, which are all measurable.

Definition 13.15 (Simple functions).

a) Consider $(\mathbb{R}^n, \mathcal{L}^n, m)$. A simple function is any $\phi : \mathbb{R}^n \to \mathbb{R}$ of the form

$$\phi = \sum_{j=1}^{\infty} c_j \chi_{A_j}$$

where $c_j \in \mathbb{R}, A_j \in \mathcal{L}^n$.

b) Let $\mathcal{S}^+(\mathbb{R}^n, \mathcal{L}^n, m) = \{\phi \text{ simple}, \phi \geq 0\}$.

c) Let $\phi \in \mathcal{S}^+$. Define

$$\int_{\mathbb{R}^n} \phi dm = \sum_{j=1}^{m} c_j m(A_j).$$

Theorem 13.16. Let $f : (\mathbb{R}, \mathcal{L}^n) \to \mathbb{R}$ be measurable, $f \geq 0$. Then, $\exists$ simple functions $\phi_n, n = 1, 2, \cdots$ s.t. $0 \leq \phi_n \nearrow f$ point-wise on $\mathbb{R}^n$.

Proposition 13.17. Suppose $\phi, \psi \in \mathcal{S}^+, c \geq 0$.

a)

$$\int_{\mathbb{R}^n} c\phi dm = c \int_{\mathbb{R}^n} \phi dm.$$

b) $\int(\phi+\psi) = \int\phi + \int\psi$.

c) $\phi \le \psi \implies \int\phi \le \int\psi$.

d) Fix ϕ. If

$$\mu(A) \equiv \int_A \phi dm \equiv \int_{\mathbb{R}^n} \phi\chi_A$$

where $A \in \mathcal{L}^n$, then μ is a measure on $\mathcal{L}^n$.

Definition 13.18 (Lebesgue integral). Let

$$f \in \mathcal{L}^n_+ \equiv \{f : \mathbb{R}^n \to \mathbb{R}, f \text{ measurable}, f \ge 0\}.$$

Define

$$\int_{\mathbb{R}^n} f dm = \sup\left\{\int_{\mathbb{R}^n} \phi dm : 0 \le \phi \le f, \phi \text{ simple}\right\}.$$

Definition 13.19 (Lebesgue integrable functions $\mathbb{L}^1$). Suppose $f : \mathbb{R}^n \to \mathbb{R}$ measurable but not necessarily ≥ 0 $\forall x$. Write $f = f^+ - f^-$ and $f^\pm$ measurable.

a) Define

$$\int_{\mathbb{R}^n} f dm = \int_{\mathbb{R}^n} f^+ dm - \int_{\mathbb{R}^n} f^- dm.$$

b) If both $\int_{\mathbb{R}^n} f^\pm dm < \infty$, say f is integrable and write

$$f \in \mathbb{L}^1(\mathbb{R}^n, \mathcal{L}^n, m).$$

Remark 13.20.

a) f integrable $\iff \int |f| dm < \infty$.

b) Let $A \in \mathcal{L}^n, f \in \mathbb{L}^1$. Define

$$\int_A f dm \equiv \int_{\mathbb{R}^n} f\chi_A dm.$$

13.5 Convergence Theorems

Theorem 13.21 (Monotone Convergence Theorem). Let $f_n \in \mathcal{L}_+^m$ be a sequence of non-negative measurable functions. Suppose f_n monotonically increasing, i.e., $f_n \leq f_{n+1}$ $\forall n$ and suppose $f(x) = \lim_{n\to\infty} f_n(x)$ point-wise. Then,

$$\lim_{n\to\infty} \int f_n dm = \int (\lim_{n\to\infty} f_n) dm = \int f dm.$$

Theorem 13.22 (Dominated Convergence Theorem). Let $f_n \in \mathbb{L}^1$. Suppose $f_n \to f$ point-wise in $\mathbb{R}^n$. (Hence, f is measurable.) Suppose $\exists g \in \mathbb{L}^1$ s.t. $|f_n| \leq g$ $\forall n$. Then, $f \in \mathbb{L}^1$ and

$$\lim_{n\to\infty} \int f_n dm = \int f dm.$$

Proposition 13.23. Suppose $f_n \in \mathcal{L}_+^n$. Then, it follows immediately from MCT that

$$\int \left(\sum_{i=1}^{\infty} f_i \right) dm = \sum_{i=1}^{\infty} \int f_i dm.$$

Proposition 13.24. Let $\phi \in \mathcal{S}^+(\mathbb{R}^n, \mathcal{L}^n, m)$ be a non-negative simple function. Then, $A \to \int_A \phi dm$ is a measure on $\mathcal{L}^m$.

Proposition 13.25 (Sets of measure 0 is negligible in Lebesgue integration theory). Let $N \in \mathcal{L}^n$, $m(N) = 0$, i.e., $\forall \epsilon > 0$, N can be covered by intervals I_j s.t.

$$\sum_{j=1}^{\infty} m(I_j) < \epsilon.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Then,

$$\int_N |f| dm = 0.$$

Corollary 13.26. Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ are in $\mathbb{L}^1$ and are hence measurable. Suppose $f = g$ except on N (say $f = g$ almost everywhere), where $N \in \mathcal{L}^n$ has $m(N) = 0$. Then,

$$\int_{\mathbb{R}^n} f - g = \int_N f - g = 0 \implies \int f = \int g.$$

Remark 13.27. Recall

$$\mathbb{L}^1(\mathbb{R}^n, \mathcal{L}^n, m) = \left\{ \text{Lebesgue measurable functions } f \text{ s.t. } \int |f| < \infty \right\}.$$

Notice that $|\cdot|$ is not a norm since we can have $\int |f| = 0$ where $f \neq 0$.

Definition 13.28. Given $f, g \in \mathbb{L}^1$, say $f \sim g \iff f = g$ almost everywhere. Define

$$L^1(\mathbb{R}^n, \mathcal{L}^n, m) = \{[f] : f \in \mathbb{L}^1\},$$

where $[f]$ denotes the equivalence class of f with the L^1 norm

$$|[f]|_{L^1} = \int |f| dm.$$

Similarly, define

$$|f|_{L^p} = \left(\int_{\mathbb{R}^n} |f|^p dm \right)^{1/p}.$$

Define

$$L^p(\mathbb{R}^n) = \{[f] : f \text{ measurable}, |f|_{L^p} < \infty\}.$$