MATH 653 Review

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Contents

1 Natural numbers, integers, rationals

Definition 1.1 (Natural numbers)**.** *We define zero and the natural numbers using sets by taking*

$$0 := \emptyset, 1 := \{\emptyset\}, 2 := \{\emptyset, \{\emptyset\}\}, \text{ etc.} \tag{1}$$

Equivalently, we can rewrite the definition as follows:

$$0 := \emptyset, 1 := \{0\}, 2 := \{0, 1\}, \text{ etc.} \tag{2}$$

Remark 1.2. *Let* $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$*. We can define* $m \leq n$ *to mean* $m \subset n$*.*

Remark 1.3 (Addition)**.** *Addition of elements* $m, n \in \mathbb{N}_0$ *can be defined as follows:*

1. *Taking the disjoint union* $m \cup n$*.*
2. *Search* $\mathbb{N}_0$ *for the unique set that can be put into one-to-one correspondence with* $m \cup n$*.*

Remark 1.4 (Multiplication)**.** *Multiplication is defined by repeated addition.*

Definition 1.5 (Integers)**.** *First, we say two ordered pairs of elements of* $\mathbb{N}_0$*,* $(m, n), (m', n')$ *are equivalent if*

$$m + n' = n + m', \tag{3}$$

in which case we write $(m, n) \sim (m', n')$*. We define the equivalence class of* (m, n)*, denoted as* $[(m, n)]$*, to be the set of all ordered pairs equivalent to* (m, n)*. Then, we can define the integer "*$m - n$*" as* $[(m, n)]$*.*

Definition 1.6 (Rationals)**.** *Assuming* $q \neq 0, q' \neq 0$*, we say to ordered pairs of elements of* $\mathbb{N}_0$*,* $(p, q), (p', q')$ *are equivalent if*

$$pq' = p'q, \tag{4}$$

in which case we write $(p, q) \sim (p', q')$*. We define the equivalence class of* (p, q)*, denoted as* $[(p, q)]$*, to be the set of all ordered pairs equivalent to* (p, q)*. Then, we can define the rational "*p/q*" as* $[(p, q)]$*.*

Remark 1.7 (Dedekind cut)**.** *For example, one can define the irrational number* π *as*

$$\pi := \{q \in \mathbb{Q} : q \leq 0\} \cup \{q \in \mathbb{Q} :\} \tag{5}$$

2 Real numbers

Definition 2.1 (Least upper bound property, completeness). *Let F be an ordered field. We say that F has the least upper bound property (or is complete) if any nonempty subset $S \subset F$ that is bounded above has a least upper bound in F.*

Theorem 2.2. *There exists a complete ordered field. We call it the real numbers and denote it by $\mathbb{R}$.*

Remark 2.3. *The complete ordered field $\mathbb{R}$ is unique, which means that if R is another complete ordered field, then there exists a bijective map $\psi : \mathbb{R} \to R$ which preserves the structure of the ordered fields $\mathbb{R}$ and R. We say ψ preserves the structure of $\mathbb{R}$ and R if for any $x, y \in \mathbb{R}$, we have*

1. $\psi(x + y) = \psi(x) + \psi(y)$, $\psi(x \cdot y) = \psi(x) \cdot \psi(y)$, *and*
2. *if* $x < y$, *then* $\psi(x) < \psi(y)$.

Corollary 2.4 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *Let $x, \epsilon \in \mathbb{R}, \epsilon > 0$. By the Archimedean property of $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $|x - r| < \epsilon$.*

Remark 2.5 ($\mathbb{Q}$ is not complete). *The ordered field $\mathbb{Q}$ is not complete.*

3 Metric spaces

Let (X, d) be a metric space and let $S \subset X$.

3.1 Open and closed sets

Definition 3.1 (Open sets). *We say S is an open set if*

$$\forall p \in S \ \exists r > 0 \ s.t. \ B(p, r) \subset S. \tag{6}$$

Definition 3.2 (Closed sets). *We say S is closed if its complement in X, $X \setminus S$ is open.*

Proposition 3.3 (Sequential characterization of closed sets). *S is closed $\iff$ $\forall p_n \in S$ s.t. $p_n \to p \in X$, we have $p \in S$.*

Definition 3.4 (Limit points)**.** *We say $p \in X$ is a limit point of S if*

$$\forall r > 0 \; \exists x \in S \setminus \{p\} \; s.t. \; x \in B(p, r) \tag{7}$$

Proposition 3.5 (Characterization of closed sets using limit points)**.** *S is closed $\iff$ S contains all its limit points.*

3.2 Completeness

Definition 3.6. *A metric space (X, d) is complete if every Cauchy sequence (p_n) in X converges to an element $p \in X$.*

Example 3.7.

1. *The metric space (X, d) where*

$$X = C([0,1], \mathbb{R}) \tag{8}$$

and

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)| \tag{9}$$

is complete.

2. *The metric space (X, d) where*

$$X = C^1([a,b], \mathbb{R}) \tag{10}$$

and

$$d(f, g) = \sup_{x \in [a,b], k=0,1} |f^{(k)}(x) - g^{(k)}(x)| \tag{11}$$

is complete.

3.3 Compactness

Definition 3.8 (Compactness)**.** *We say $K \subset X$ is compact if any open cover of K can be reduced to a finite subcover.*

Definition 3.9 (Sequential compactness)**.** *We say $K \subset X$ is sequentially compact if any sequence in K has a subsequence that converges to a point of K.*

Theorem 3.10. *A set $K \subset (X, d)$ is compact $\iff$ it is sequentially compact.*

Definition 3.11 (Total boundedness)**.** *A metric space (X, d) is totally bounded if $\forall \epsilon > 0$, X is the union of of a finite number of open balls of radius ϵ.*

Proposition 3.12. *The following are equivalent:*

1. *(X, d) is compact,*
2. *(X, d) is sequentially compact, and*
3. *(X, d) is complete and totally bounded.*

Proposition 3.13. *In any metric space (X, d) if $K \subset X$ is compact, then K is closed and bounded.*

Remark 3.14. *The converse of the previous proposition is not true in a general metric space.*

Theorem 3.15 (Heine-Borel theorem)**.** *In $(\mathbb{R}^n, |x - y|)$, any closed and bounded set is compact.*

Example 3.16. *The closed unit ball in $C([0, 1], \mathbb{R})$ equipped with the usual sup norm is not compact. Indeed, consider the sequence of functions (f_n) given by*

$$f_n(x) = x^n. \tag{12}$$

If f_n were to converge, then f_n would converge uniformly to some $f \in C([0, 1], \mathbb{R})$. But we know f_n converges to g point-wise where

$$g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}, \tag{13}$$

which is discontinuous at 1.

3.4 Connectedness

Definition 3.17 (Connectedness)**.**

1. *We say a metric space (X, d) is connected if X cannot be written as the union of two disjoint, nonempty, open sets.*

2. *If $S \subset X$, we say S is connected if the metric space (S, d) is connected.*

Proposition 3.18. *The metric space (X, d) is connected if and only if the only subsets of X that are both open and closed are X and the empty set $\emptyset$.*

Definition 3.19 (Path-connectedness)**.** *We say X is path connected if (X, d) has the property that for any $p, q \in X$, there exists a continuous map $\gamma : [0, 1] \to X$ with $\gamma(0) = p, \gamma(1) = q$.*

Proposition 3.20. *Any path-connected metric space is connected.*

3.5 Contraction mapping theorem

Definition 3.21 (Contraction)**.** *A map $\phi : X \to X$ is a contraction if $\exists c \in (0, 1)$ s.t.*

$$d(\phi(x) - \phi(y)) \leq cd(x, y) \ \forall x, y \in X. \tag{14}$$

Theorem 3.22 (Contraction mapping theorem)**.** *Let (X, d) be a nonempty and complete metric space. Suppose $\phi : X \to X$ is a contraction. Then $\exists$ a unique $x \in X$ s.t. $\phi(x) = x$ and we call x a fixed point.*

3.6 Tricks and examples

Example 3.23 (Closed and bounded but not compact)**.** *The closed unit ball $B \subset C([0, 1], \mathbb{R})$ equipped with the usual sup norm is closed and bounded in $C([0, 1], \mathbb{R})$ but not compact. Suppose B is compact. Consider the sequence of functions (f_n) given by $f_n(x) = x^n$. Then, $\exists$ subsequence $f_{n_k} \to g \in C([0, 1], \mathbb{R})$. But we already know that f_n converges pointwise to f given by*

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}, \tag{15}$$

which has discontinuity at 1, which is contradiction.

Proposition 3.24. *Any compact metric space is complete.*

Proof. Let (X, d) be a compact metric space. Let (x_n) be a Cauchy sequence in X. Fix $\epsilon > 0$.

1. Since X compact, we know $\exists$ subsequence $x_{n_k} \to x \in X$, i.e., $\exists N_1 \in \mathbb{N}$ s.t.

$$d(x_{n_k}, x) < \epsilon/2 \ \forall k \geq N_1. \tag{16}$$

2. Since (x_n) Cauchy, we know $\exists N_2 \in \mathbb{N}$ s.t.

$$d(x_m, x_n) < \epsilon/2 \ \forall m, n \geq N_2. \tag{17}$$

3. Hence, take $N = \max\{N_1, N_2\}$, then $\forall k \geq N$, we also have $n_k \geq k \geq N$, in which case

$$d(x_k, x) \leq d(x_k, x_{n_k}) + d(x_{n_k}, x) < \epsilon/2 + \epsilon/2 = \epsilon. \tag{18}$$

□

4 Continuous function on metric spaces

4.1 Continuity

Proposition 4.1 (Continuity)**.** *The following properties of $f : X \to Y$ are equivalent:*

1. *$x_n \to a$ implies $f(x_n) \to f(a)$,*
2. *$\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $d(x, a) < \delta$ implies $d(f(x), f(a)) < \epsilon$.*
3. *If $\mathcal{O}$ is any open set containing $f(a)$, then the preimage $f^{-1}(\mathcal{O})$ contains $B(a, \delta)$ for some $\delta > 0$.*

Proposition 4.2. *Let $f : X \to Y$. Then f is continuous if and only if for any open set $\mathcal{O} \subset Y$ the preimage $f^{-1}(\mathcal{O})$ is open in X.*

Definition 4.3 (Uniform continuity)**.** *Let $f : X \to Y$. We say f is uniformly continuous on X if $\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $d(x_1, x_2) < \delta$ implies*

$$d(f(x_1), f(x_2)) < \epsilon \ \forall x_1, x_2 \in X. \tag{19}$$

Proposition 4.4. *Let (X, d) be compact and suppose $f : X \to Y$ is continuous. Then f is uniformly continuous on X.*

Remark 4.5. *A continuous function on a compact set $K \subset X$ is uniformly continuous on X.*

Proposition 4.6 (Failure of uniform continuity)**.** *Let $f : X \to Y$. Then f fails to be uniformly continuous if and only if there exists $\epsilon > 0$ and sequences $(p_n), (q_n)$ in X s.t. $d(p_n, q_n) \to 0$ as $n \to \infty$ but $d(f(p_n), f(q_n)) \geq \epsilon \ \forall n$.*

4.2 Extreme value theorem

Proposition 4.7. *Let (X, d) be compact and suppose $f : X \to Y$ is continuous. Then $f(X)$ is compact.*

Corollary 4.8 (Extreme value theorem)**.** *Let (X, d) be compact and suppose $f : X \to \mathbb{R}$ is continuous. Then f attains an absolute max and an absolute min on X.*

Proposition 4.9. *Let $f : X \to Y$ be continuous. If (X, d) is connected, then $f(X)$ is connected.*

4.3 Intermediate value theorem

Corollary 4.10 (Intermediate value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f assumes every value between $f(a)$ and $f(b)$.*

4.4 Sequences of functions

Definition 4.11 (Convergence of sequences of functions)**.**

1. *The sequence (f_n) converges pointwise to f if given any $x \in X$ and $\epsilon > 0$, there exists $N = N(x, \epsilon) \in \mathbb{N}$ s.t. $n \geq N$ implies $d(f_n(x), f(x)) < \epsilon$.*

2. *The sequence (f_n) converges uniformly to f if given any $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ s.t. $n \geq N$ implies $d(f_n(x), f(x)) < \epsilon \ \forall x \in X$.*

3. *The sequence (f_n) is uniformly Cauchy on X if given any $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ s.t. $n \geq N$ implies $d(f_n(x), f_m(x)) < \epsilon \ \forall x \in X$.*

Proposition 4.12. *Suppose $f_n : X \to Y$ are continuous and $f_n \to f$ uniformly on X. Then, $f : X \to Y$ is continuous.*

Proposition 4.13 (Failure of uniform convergence)**.** *Consider a sequence of functions $f_n : X \to Y$ and a function $f : X \to Y$ such that $f_n \to f$ pointwise. Suppose that there exists $\epsilon > 0$, $N > 0$, and a sequence (x_n) in X s.t. $d(f_n(x_n), f(x_n)) \geq \epsilon \ \forall n \geq N$. Then, uniform convergence fails.*

Proposition 4.14.

1. *Suppose $f_n : X \to \mathbb{R}$ is a uniformly Cauchy sequence of functions. There exists a function $f : X \to \mathbb{R}$ s.t. $f_n \to f$ uniformly on X.*

2. *If the f_n are continuous, so is f.*

Proposition 4.15.

1. *Suppose $f_n : X \to Y$ is a uniformly Cauchy sequence of functions. There exists a function $f : X \to Y$ s.t. $f_n \to f$ uniformly on X.*

2. *If the f_n are continuous, so is f.*

Proposition 4.16 (Interchanging limit processes)**.**

1. *Suppose $f_n : X \to Y$ are continuous on X and $f_n \to f$ uniformly on X. Then, $f : X \to Y$ is continuous.*

$$\lim_{x\to a} f(x) = \lim_{x\to a}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to a} f_n(x) = \lim_{n\to\infty} f_n(a) = f(a). \quad (20)$$

2. *Suppose $f_n : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and $f_n \to f$ uniformly on $[a,b]$. Then, f is continuous on $[a,b]$ and*

$$\lim_{n\to\infty}\int_a^b f_n(t)\ dt = \int_a^b f(t)\ dt. \quad (21)$$

3. *Let $I \subset \mathbb{R}$ be an open interval and suppose $f_n : I \to \mathbb{R}$ are C^1 functions. Let $f, g : I \to \mathbb{R}$ and suppose that given any compact subset $K \subset I$, the sequences f_n and f_n' converge uniformly on K to f and g respectively. Then, f is C^1 and $f' = g$, i.e.,*

$$\left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty} f_n'. \quad (22)$$

4.5 Tricks and examples

Proposition 4.17. *Consider a sequence of functions $f_n : X \to Y$ and a function $f : X \to Y$ s.t. $f_n \to f$ pointwise. Then uniform convergence fails if and only if there exists an $\epsilon > 0$, a sequence (x_n) in X, and a subsequence (f_{n_k}) of (f_n) s.t.*

$$d(f_{n_k}(x_k), f(x_k)) \geq \epsilon\ \forall k. \quad (23)$$

Proof.

1. [$\Longrightarrow$] Since uniform convergence fails, we know $\exists \epsilon > 0$ s.t. $\forall N \in \mathbb{N}\ \exists n \geq N$ s.t. $\exists x \in X$ s.t. $|f_n(x) - f(x)| \geq \epsilon$. For convenience, let $n_0 = 0$. Having chosen n_{k-1}, we can pick $n_k \geq n_{k-1} + 1$ s.t. $\exists x_k$ s.t. $|f_{n_k}(x_k) - f(x_k)| \geq \epsilon$.

2. [$\Longleftarrow$] For the sake of contradiction, assume $f_n \to f$ uniformly and so does f_{n_k}. In particular, for the given $\epsilon > 0$, we know $\exists N \geq \mathbb{N}$ s.t. $\forall n \geq N$, $d(f_{n_k}(x) - f(x)) < \epsilon\ \forall x$, which is contradiction.

□

Remark 4.18. *The metric space $X = C^1([a, b], \mathbb{R})$ equipped with the norm*

$$|f| = \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)| \tag{24}$$

is complete.

Proof. Suppose $f_n \in X$ is Cauchy. Then $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N$, we have

$$\sup_{x \in [a,b]} |f_m(x) - f_n(x)| + \sup_{x \in [a,b]} |f'_m(x) - f'_n(x)| < \epsilon. \tag{25}$$

Hence, f_n and f'_n are both uniformly Cauchy on $[a, b]$ and thus converges uniformly on $[a, b]$, i.e., $f_n \to f$ and $f'_n \to g$ uniformly. Notice that by uniform convergence, we know $g = f'$.

Fix $\epsilon > 0$. Since $f_n \to f$ uniformly, we know $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| < \epsilon/2. \tag{26}$$

Since $f'_n \to f'$ uniformly, we know $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$\sup_{x \in [a,b]} |f'_n(x) - f'(x)| < \epsilon/2. \tag{27}$$

Hence, pick $N = \max\{N_1, N_2\}$, in which case, $\forall n \geq N$, we have

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| + \sup_{x \in [a,b]} |f'_n(x) - f'(x)| < \epsilon/2 + \epsilon/2 = \epsilon. \tag{28}$$

□

5 Differentiability

5.1 Differentiability

Definition 5.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$. We say f is differentiable at $a \in \mathbb{R}^n$ if $\exists c \in \mathbb{R}^n$ s.t. the function r defined by*

$$f(a+h) = f(a) + c \cdot h + r(h) \tag{29}$$

satisfies

$$\lim_{h \to 0} \frac{r(h)}{|h|} = 0. \tag{30}$$

Definition 5.2. *Let $F = (f_1, \cdots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$. We say F is differentiable at $a \in \mathbb{R}^n$ if $\exists C \in \mathcal{M}_{m \times n}$ s.t. the function defined by*

$$F(a+h) = F(a) + C \cdot h + r(h) \tag{31}$$

satisfies

$$\lim_{h \to 0} \frac{r(h)}{|h|} = 0. \tag{32}$$

Theorem 5.3 ("A simple criterion")**.** *Let $\mathcal{O} \subset \mathbb{R}^n$ and $f : \mathcal{O} \to \mathbb{R}$. Suppose $f \in C^1(\mathcal{O}, \mathbb{R})$. Then, f is differentiable at any $x \in \mathcal{O}$.*

5.2 Chain rule

Theorem 5.4 (Chain rule)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $x \in \mathbb{R}^n$. Let $G : \mathbb{R}^m \to \mathbb{R}^k$ be differentiable at $z \equiv F(x)$. Then $H = G \circ F : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable at x and*

$$DH(x) = DG(F(x)) \cdot DF(x) \tag{33}$$

where

$$D(F(a)) = \begin{pmatrix} \nabla f_1(a) \\ \vdots \\ \nabla f_m(a) \end{pmatrix}. \tag{34}$$

5.3 Clairaut's theorem

Theorem 5.5 (Clairaut's theorem)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be C^2. Then*

$$\partial_j \partial_k F(x) = \partial_k \partial_j F(x) \ \forall x.$$

5.4 Tricks and examples

Lemma 5.6 ("FTC lemma"). *Let $f : (a,b) \to \mathbb{R}$ be C^1. Then,*

$$f(x+y) - f(x) = \left(\int_0^1 f'(x+ty)\, dt \right) y. \tag{35}$$

Compare the lemma above to mean value theorem.

Theorem 5.7 (Mean value theorem). *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then, $\exists c$ between x and $x+y$ s.t.*

$$f(x+y) - f(x) = f'(c)y. \tag{36}$$

Example 5.8 (Standard pathological example). *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}. \tag{37}$$

Notice that $f_x(0,0) = f_y(0,0) = 0$ but f is not continuous or differentiable at $(0,0)$.

6 Taylor's theorem

6.1 Multi-index notation

Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$.

Definition 6.1. *Multi-index*

1. *A multi-index is an n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ where $\alpha_j \in \mathbb{N}_0$.*
2. *Define $x^\alpha := x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. In addition, define $x_j^0 := 1$ even if $x_j = 0$.*
3. *The order of α is $|\alpha| := \alpha_1 + \cdots + \alpha_n$.*
4. *Define $\alpha! := \alpha_1! \alpha_2! \cdots \alpha_n!$. In addition, define $0! := 1$.*

Remark 6.2 (Polynomials). *Any polynomial $p(x)$ of order $\leq m$ can be written as*

$$p(x) = \sum_{|\alpha| \leq m} c_\alpha x^\alpha \quad \text{where } c_\alpha \text{ constant.} \tag{38}$$

Definition 6.3. *Let* $\alpha = (\alpha_1, \cdots, \alpha_n)$ *be a multi-index and let*

$$\partial = (\partial_{x_1}, \cdots, \partial_{x_n}) = (\partial_1, \cdots, \partial_n). \tag{39}$$

We define

$$\partial^\alpha = \partial_1^{\alpha_1} \circ \cdots \circ \partial_n^{\alpha_n}. \tag{40}$$

6.2 Multinomial theorem

Theorem 6.4 (Binomial theorem)**.**

$$(x_1 + x_2)^m = \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} x_1^{m-j} x_2^j. \tag{41}$$

Theorem 6.5 (Multinomial theorem)**.**

$$(x_1 + \cdots + x_n)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha. \tag{42}$$

6.3 Taylor's theorem

Theorem 6.6 (Taylor's theorem)**.** *Let* $m \in \mathbb{N}$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *and suppose* $f \in C^{m+1}$. *Let* $a, x \in \mathbb{R}$. *Then,*

$$f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(m+1)}(\xi)}{(m+1)!}(x-a)^{m+1}, \tag{43}$$

where ξ *is strictly between* a, x.

Theorem 6.7 (Taylor's theorem)**.** *Let* $m \in \mathbb{N}$. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *and suppose* $f \in C^{m+1}$. *Let* $a, x \in \mathbb{R}^n$. *Then,*

$$f(x) = \sum_{|\alpha| \le m} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + \sum_{|\alpha|=m+1} \frac{D^\alpha f(\xi)}{\alpha!}(x-a)^\alpha, \tag{44}$$

where ξ *is strictly between* a, x, *i.e.,* ξ *lies on the open segment joining* a, x.

7 Limit superior and limit inferior

Definition 7.1. *Let* (a_n) *be any sequence in* $\mathbb{R}$*, we define*

$$\limsup_{n\to\infty} a_n = \lim_{m\to\infty} \sup_{n\geq m} a_n, \tag{45}$$

$$\liminf_{n\to\infty} a_n = \lim_{m\to\infty} \inf_{n\geq m} a_n. \tag{46}$$

Proposition 7.2. *Let* (a_n) *and* (b_n) *be sequences in* $\mathbb{R}$*, then*

1. $\limsup(-a_n) = \liminf a_n$,
2. $\limsup(ca_n) = c\limsup a_n$ *for any* $c > 0$,
3. $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$,
4. $\liminf a_n \leq \limsup a_n$ *where equality holds if and only if* (a_n) *converges, in which case* $\liminf a_n = \limsup a_n = \lim a_n$*, and*
5. *if* (b_n) *is a subsequence of* (a_n)*, then*

$$\liminf a_n \leq \liminf b_n \leq \limsup b_n \leq \limsup a_n. \tag{47}$$

8 Contraction mapping theorem

Theorem 8.1 (Contraction mapping theorem)**.** *Let* (X, d) *be a nonempty, complete metric space. Suppose* $f : X \to X$ *has the following property:* $\exists k$ *with* $0 \leq k < 1$ *s.t.*

$$d(f(x), f(y)) \leq kd(x, y) \quad \forall x, y \in X. \tag{48}$$

Then, f *has a unique fixed point in* X*.*

9 Inverse and implicit function theorems

9.1 Inverse function theorem

Theorem 9.1 (Inverse function theorem)**.** *Suppose* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is* C^1 *on* $\mathbb{R}^n$*. Let* $a \in \mathbb{R}^n$*. We have the following.*

1. *If $f'(a) \in \mathcal{M}_{n\times n}$ is invertible, then $\exists$ open sets $U \ni a$ and $V \ni f(a) = b$ s.t. $f : U \to V$ is a C^1-diffeomorphism, i.e., f is one-to-one, onto, and both f and f^{-1} are C^1.*

2. *Let $g = f^{-1} : V \to U$ then g is C^1 and*

$$g'(f(x)) = [f'(x)]^{-1}\ \forall x \in U. \tag{49}$$

Theorem 9.2 (Inverse function theorem)**.** *Let V be a finitely dimensional real normed vector space. Suppose $f : V \to V$ is C^1 on V. Let $a \in V$.*

Then if $f'(a) \in L(V,V)$ is invertible, then $\exists$ open sets $U_1 \ni a$ and $U_2 \ni f(a) = b$ s.t. $f : U_1 \to U_2$ is a C^1-diffeomorphism.

9.2 Implicit Function Theorem

Theorem 9.3 (Implicit function theorem)**.** *Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be C^1. In addition, we write $f(x,y)$ with $x \in \mathbb{R}^n, y \in \mathbb{R}^m$. Suppose $f(a,b) = 0$ and assume $D_x f(a,b) =: A_x$ is invertible. Then*

1. *$\exists$ open sets $U \ni (a,b)$ in $\mathbb{R}^{n+m}$ and open sets $W \ni b$ in $\mathbb{R}^m$ and C^1 function $g : W \to \mathbb{R}^n$ s.t.*

$$\{(x,y) \in U : f(x,y) = 0\} = \{(g(y), y) : y \in W\}. \tag{50}$$

2. *If $A_y = D_y f(a,b)$, then $g'(b) = -A_x^{-1} A_y$.*

9.3 Lagrange multipliers

Proposition 9.4 (Lagrange multiplier)**.** *Let $f, g \in C^1(\mathbb{R}^3, \mathbb{R})$. Let $S = \{x \in \mathbb{R}^3 : g(x) = 0\}$. Let $a \in S$ and assume $\nabla g(a) \neq 0$. If $f|_S$ has a local maximum at $a \in S$, then $\exists \lambda \in \mathbb{R}$ s.t.*

$$\nabla f(a) = \lambda g(a). \tag{51}$$

10 Partition of unity

Proposition 10.1 (Partition of unity)**.** *Let $K \subset \mathbb{R}^n$ be compact and suppose $\{U_j, j = 1, \cdots, N\}$ is an open cover of K. A C^∞ partition of unity on K subordinate to this covering is a collection $\{\rho_j, j = 1, \cdots, N\}$ of C^∞ functions $\rho_j : \mathbb{R}^n \to \mathbb{R}$ with the properties*

1. $\operatorname{supp} \rho_j \subset u_j$ *for every* j, *and*

2. $\sum_{j=1}^{N} \rho_j = 1$ *on* K.

11 Basics of measure theory

11.1 $\sigma-$algebra

Definition 11.1 ($\sigma-$algebra). *Let* X *be a nonempty set. We say* $\mathcal{A} \subset \mathcal{P}(X)$ *is* $\sigma-$*algebra on* X *if* $\mathcal{A}$ *is closed under countable unions and taking complements, i.e.,*

1. *if* $E_1, E_2, \cdots \in \mathcal{A}$, *then* $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$, *and*

2. *if* $E \in \mathcal{A}$, *then* $E^c := X \setminus E \in \mathcal{A}$.

Remark 11.2. *If* $E_1, E_2, \cdots \in \mathcal{A}$, *then* $\bigcap_{j=1}^{\infty} E_j \in A$.

Corollary 11.3. *If* $\mathcal{E} \subset \mathcal{P}(X)$. *Then there is a unique smallest* $\sigma-$*algebra that contains* $\mathcal{E}$, $\sigma(\mathcal{E})$. *Call it the* $\sigma-$*algebra generated by* $\mathcal{E}$, *where*

$$\sigma(\mathcal{E}) = \bigcap \{\sigma - \textit{algebra that contain } \mathcal{E}\}. \tag{52}$$

Definition 11.4 (Borel $\sigma-$algebra). *We define the Borel* $\sigma-$*algebra* $\mathcal{B}_X$ *on* X *to be the* $\sigma-$*algebra generated by the set of all open sets in* X, *i.e.,*

$$\mathcal{B}_X = \sigma(\{\textit{open sets in } X\}). \tag{53}$$

11.2 Measure

Let X be a nonempty set. Let $\mathcal{M}$ be a $\sigma-$algebra on X.

Definition 11.5 (Measure). *A measure* μ *on* $(X, \mathcal{M})$ *is a function* $\mu : \mathcal{M} \to [0, \infty]$ *s.t.*

1. $\mu(\emptyset) = 0$, *and*

2. *(Countable additivity) if* $E_j \in \mathcal{M}, j = 1, 2, \cdots$ *disjoint, then*

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j). \tag{54}$$

We call $(X, \mathcal{M}, \mu)$ a measure space.

Proposition 11.6 (Properties of measures)**.**

1. *(Monotonocity). Let $E, F \in \mathcal{M}$. Then $E \subset F \implies \mu(E) \leq \mu(F)$.*
2. *(Subadditivity). Let $E_1, E_2, \dots \in \mathcal{M}$ not necessarily disjoint, then*

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j). \tag{55}$$

3. *(Continuity from below). Let $E_1 \subset E_2 \subset \cdots$ where $E_j \in \mathcal{M}$. Then,*

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j). \tag{56}$$

4. *(Continuity from above). Let $E_1 \supset E_2 \supset \cdots$ where $E_j \in \mathcal{M}$. Assume $\mu(E_1) < \infty$. Then,*

$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j). \tag{57}$$

Definition 11.7 (Outer measure)**.** *An outer measure on set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ s.t.*

1. *$\mu^*(\emptyset) = 0$,*
2. *(Monotonocity) $A \subset B \implies \mu^*(A) \leq \mu^*(B)$, and*
3. *(Subadditivity) if $A_j \in \mathcal{P}(X)$, then*

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j). \tag{58}$$

Definition 11.8 (μ^*-measurable)**.** *Let μ^* be an outer measure on X. Let $A \subset X$. We say A is μ^*-measurable if for every subset $E \subset X$, we have*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c). \tag{59}$$

Theorem 11.9 (Caratheodory I)**.** *Let μ^* be an outer measure on X. The collection $\mathcal{M} \subset \mathcal{P}(X)$ of μ^*-measurable sets is a $\sigma-$algebra and $\mu^*|_{\mathcal{M}} =: \mu$ is a measure of X and $(X, \mathcal{M}, \mu)$ is a measure space.*

Definition 11.10 (Metric outer measure)**.** *An outer measure μ^* on X is a metric outer measure if whenever $S_1, S_2 \subset X$ and*

$$d(S_1, S_2) = \inf\{d(x_1, x_2) : x_i \in S_i\} > 0, \tag{60}$$

we have

$$\mu^*(S_1 \cup S_2) = \mu^*(S_1) + \mu^*(S_2). \tag{61}$$

Theorem 11.11 (Caratheodory II)**.** *If μ^* is a metric outer measure on X, then every closed subset in X is μ^*-measurable.*

11.3 Lebesgue measure

Definition 11.12 (Lebesgue outer measure)**.** *Let*

$$\mathcal{E} = \{\textit{bounded open intervals in } \mathbb{R}^n\}. \tag{62}$$

An open interval $I \in \mathcal{E}$ has the form

$$I = \{x \in \mathbb{R}^n : a_i < x_i < b_i, a_i, b_i \in \mathbb{R}^n\}. \tag{63}$$

Let $\lambda : \mathcal{E} \to [0, \infty]$ be given by

$$\lambda(I) = \prod_{j=1}^{n} (b_j - a_j). \tag{64}$$

If $S \subset \mathbb{R}^n$, let

$$m^*(S) := \inf\left\{\sum_{j=1}^{\infty} \lambda(I_j) : S \subset \bigcup_{j=1}^{\infty} I_j, I_j \in \mathcal{E}\right\}. \tag{65}$$

Definition 11.13 (Lebesgue measurable sets)**.** *We define*

$$\mathcal{L}_n = \{m^* - \textit{measurable sets on } \mathbb{R}^n\}. \tag{66}$$

By Caratheodory I, we define $m := m^*|_{\mathcal{L}_n}$ to be the Lebesgue measure of $\mathbb{R}^n$.

Theorem 11.14 (Regularities of Lebesgue measure)**.** *Let $B \in \mathcal{L}_n$, then*

$$m(B) = \sup\{m(K) : K \subset B, K \textit{ compact}\} \tag{67}$$

$$= \inf\{m(U) : B \subset U, U \textit{ open}\}. \tag{68}$$

11.4 Complete measure space

Definition 11.15. *A measure μ on $(X, \mathcal{M})$ is complete if*

$$A \in \mathcal{M}, \mu(A) = 0, S \subset A \implies S \in \mathcal{M}, \mu(S) = 0. \tag{69}$$

Proposition 11.16. *Suppose μ is a measure on $(X, \mathcal{F})$ that is not complete. Then,*

1. $\overline{\mathcal{F}} := \{E \cup S : E \in \mathcal{F}, S \subset F \in \mathcal{F}, \mu(F) = 0\}$ *is a $\sigma-$algebra, and*
2. $\overline{\mu}(E \cup S) := \mu(E)$ *is complete.*

12 The Lebesgue integral

12.1 Lebesgue integration

Definition 12.1 (Measurable functions)**.** *Let $\mathcal{M}, \mathcal{N}$ be $\sigma-$algebras on sets X, Y, respectively. Then, $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is $(\mathcal{M}, \mathcal{N})-$measurable if $f^{-1}(\mathcal{N}) \subset \mathcal{M}$.*

Remark 12.2.

1. *If $f : (X, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then f is measurable if $f^{-1}(\mathcal{B}(\mathbb{R})) \subset \mathcal{M}$.*
2. *To check f is measurable, it is enough to check $f^{-1}(a, \infty) \in \mathcal{M}$ since (a, ∞) generate $\mathcal{B}(\mathbb{R})$.*

Definition 12.3 (Simple functions)**.** *A function $f : X \to \mathbb{R}$ is simple if it assumes only finitely many distinct values.*

Remark 12.4.

1. *If $c_1, \cdots, c_n$ are the distinct values, we can write*

$$f = \sum_{j=1}^{n} c_j \chi_{A_j} \quad \text{where } A_j = \{x \in X : f(x) = c_j\}. \tag{70}$$

We call this the "canonical" or "standard" representation of f.

2. *The set $X = \bigcup_{j=1}^{n} A_j$ is a disjoint union.*

3. *Let $f : (X, \mathcal{M}) \to \mathbb{R}$ be simple. Then, f is measurable if and only if each $A_j \in \mathcal{M}$.*

Definition 12.5. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{S}^+(X, \mathcal{M}, \mu)$ be the set of non-negative measurable simple functions on $(X, \mathcal{M})$.*

1. *If*
$$\phi = \sum_{j=1}^{n} c_j \chi_{A_j} \in S^+, \tag{71}$$
define
$$\int_X \phi \; d\mu = \sum_{j=1}^{n} c_j \mu(A_j). \tag{72}$$

2. *If $A \in \mathcal{M}$, define*
$$\int_A \phi \; d\mu = \int_X \phi \chi_A \; d\mu \tag{73}$$

Theorem 12.6 (Ladder theorem)**.** *Let $f : (X, \mathcal{M}) \to (\overline{\mathbb{R}}^+, \mathcal{B}(\overline{\mathbb{R}}))$ be measurable and non-negative. Then, there exists simple functions $\phi_n \in S^+$ s.t. $0 \leq \phi_n \nearrow f$ pointwise on X.*

Remark 12.7.

1. *On any $B \in \mathcal{M}$ where f is bounded, $\phi_n \nearrow f$ uniformly on B.*
2. *Let $f : (X, \mathcal{M}) \to \overline{\mathbb{R}}$. Write $f = f^+ - f^-$. Set*
$$\phi_n(x) = \phi_n^+(x) - \phi_n^-(x) \quad \textit{where } \phi_n^{\pm} \nearrow f^{\pm}. \tag{74}$$
Notice that $\phi_n \to f$ pointwise on X.
3. *Let $f : (X, \mathcal{M}) \to \mathbb{C}$ where $f = g + ih$. Apply above to g, h, we get*
$$\psi_n + i\zeta_n \to f \textit{ pointwise on } X. \tag{75}$$

Definition 12.8. *Let $\mathcal{M}^+$ be the set of non-negative measurable functions $f : X \to \overline{\mathbb{R}}$ on $(X, \mathcal{M}, \mu)$. Let $f \in \mathcal{M}^+$. Define*
$$\int_X f \; d\mu = \sup \left\{ \int \phi \; d\mu : 0 \leq \phi \leq f, \textit{ where } \phi \textit{ is simple and measurable} \right\}. \tag{76}$$

Definition 12.9.

1. *Let* $f : X \to \overline{\mathbb{R}}$ *be measurable. Write* $f = f^+ - f^-$ *where* $f^\pm \in \mathcal{M}^+$*. Define*

$$\int f \; d\mu = \int f^+ \; d\mu - \int f^- \; d\mu, \tag{77}$$

provided at least one of $\int f^\pm$ *is finite.*

2. *If both* $\int f^\pm < \infty$*, we say* f *is integrable and write* $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$*.*

Remark 12.10. *Since*

$$f^\pm \leq |f| = f^+ + f^-, \tag{78}$$

we see that f *is integrable if and only if* $\int |f| \; d\mu < \infty$*.*

Definition 12.11.

1. *Let* $f : X \to \mathbb{C}$ *be measurable. Write* $f = \operatorname{Re} f + \operatorname{Im} f$*. Define*

$$\int f \; d\mu = \int \operatorname{Re} f \; d\mu + i \int \operatorname{Im} f \; d\mu. \tag{79}$$

2. *If* $\operatorname{Re} f, \operatorname{Im} f \in \mathcal{L}^1$*, we say* f *is integrable and write* $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$*.*

Remark 12.12. *Since*

$$|f| \leq |\operatorname{Re} f| + |\operatorname{Im} f| \leq 2|f|, \tag{80}$$

we see that f *is integrable if and only if* $\int |f| \; d\mu < \infty$*.*

12.2 Convergence theorems

Theorem 12.13 (Monotone convergence theorem)**.** *Let* $f_n \leq f_{n+1}, f_n \in \mathcal{M}^+ \; \forall n$*. Let* f *be the pointwise limit of* f_n*. Then,*

$$\lim_{n\to\infty} \int f_n \; d\mu = \int f \; d\mu. \tag{81}$$

Lemma 12.14 (Fatou's lemma)**.** *Let* $f_n \in \mathcal{M}^+ \; \forall n$*. Then,*

$$\int \left(\liminf_{n\to\infty} f_n \right) \; d\mu \leq \liminf_{n\to\infty} \int f_n \; d\mu. \tag{82}$$

Theorem 12.15 (Dominated convergence theorem)**.** *Let* $f_n \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, $f_n : X \to \mathbb{R}$. *Let* $f : X \to \mathbb{R}$ *and assume* $f_n \to f$ *pointwise* $\forall x \in X$. *Assume there exists* $g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ *s.t.* $|f_n| \leq g$ $\forall n$ *on* X. *Then,* $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ *and*

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu. \tag{83}$$

Remark 12.16. *In the monotone convergence theorem, Fatou's lemma, and the dominated convergence theorem, the hypothesis that* $f_n \to f$ $\forall x \in X$ *can be weakened to* $f_n \to f$ *a.e. on* X.

12.3 L^p spaces

Definition 12.17 ($\mathcal{L}^p$)**.** *For* $p \in [1, \infty)$, *we define*

$$\mathcal{L}^p(X, \mathcal{M}, \mu) = \left\{ f : f \text{ measurable}, \int_X |f|^p \, d\mu < \infty \right\}. \tag{84}$$

Definition 12.18 (L^p)**.** *For* $p \in [1, \infty)$, *we define* $L^p(X, \mathcal{M}, \mu)$ *to be the equivalence classes of elements of* $\mathcal{L}^p$ *where* $f \sim g \iff f = g$ *a.e.*

Definition 12.19 (Norm on L^p)**.** *We define the norm on* L^p *by*

$$|[f]|_{L^p} = \left(\int_X |f|^p \, d\mu \right)^{1/p}, \tag{85}$$

where f *is any representative of* $[f]$.

Definition 12.20 ($\mathcal{L}^\infty$)**.** *We define* $\mathcal{L}^\infty(X, \mathcal{M}, \mu)$ *to be the set of measurable functions* f *s.t.* $\exists M$ *s.t.* $|f| \leq M$ *a.e. on* X.

Definition 12.21 (L^∞)**.** *We define* $L^p(X, \mathcal{M}, \mu)$ *to be the equivalence classes of elements of* $\mathcal{L}^\infty$.

Definition 12.22 (Norm on L^∞)**.** *We define the norm on* L^∞ *by*

$$\begin{aligned} |f|_{L^\infty} &= \inf \left\{ \sup_X |g| : g \in [f] \right\} &(86) \\ &= \inf \left\{ M : \mu\{x : f(x) > M\} = 0 \right\}. &(87) \end{aligned}$$

Proposition 12.23 (Minkowski's inequality). *Let $p \in [1, \infty]$. For any measurable functions v, w, we have*

$$|v + w|_{L^p} \le |v|_{L^p} + |w|_{L^p}. \tag{88}$$

Proposition 12.24 (Hölder's inequality). *Let $p \in [1, \infty]$. Define q by $1/p + 1/q = 1$. For any measurable functions f, g, we have*

$$|fg|_{L^1} \le |f|_{L^p} |g|_{L^q}. \tag{89}$$

Theorem 12.25. *For $p \in [1, \infty]$, the normed vector space $L^p(X, \mathcal{M}, \mu)$ is complete.*

Corollary 12.26. *For $p \in [1, \infty)$, if $g_k \to g$ in $L^p(X, \mathcal{M}, \mu)$, then $\exists$ subsequence that converges pointwise $\mu-$a.e. to g.*

Proposition 12.27. *For $p \in [1, \infty)$, $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.*

12.4 Tonelli's and Fubini's theorems

Theorem 12.28 (Tonelli's theorem). *We write $\mathbb{R}^n = \mathbb{R}^k_x \times \mathbb{R}^l_y$ with $k + l = n$. Let $(x, y) \in \mathbb{R}^n$. Given $f(x, y)$, we set $f_x(y) = f^y(x) = f(x, y)$.*

Let $f \ge 0$ be Borel measurable on $\mathbb{R}^n$. Then, the functions

$$g(x) = \int f_x(y) \; dm(y) \tag{90}$$

$$h(y) = \int f^y(x) \; dm(x) \tag{91}$$

are Borel mesurable and

$$\int_{\mathbb{R}^n} f \; dm(x, y) = \int_{\mathbb{R}^k} g(x) dm(x) = \int_{\mathbb{R}^l} h(y) dm(y). \tag{92}$$

Theorem 12.29 (Fubini's theorem). *Let f be Borel measurable on $\mathbb{R}^n$ and $\int |f| \; dm < \infty$. Then, f_x is integrable for a.e. x and f^y is integrable for a.e. y, and*

$$\int f \; dm = \iint f(x, y) \; dm(y) dm(x) \tag{93}$$

$$= \iint f(x, y) \; dm(x) dm(y). \tag{94}$$

Proposition 12.30. *Suppose f is Lebesgue measurable on $\mathbb{R}^n$. Then, there exists a Borel measurable function g s.t. $f = g$ a.e. with respect to the Lebesgue measure m.*

Remark 12.31 (Fubini's theorem for Lebesgue measurable functions)**.** *Let $f \in L^1(\mathbb{R}^n, m)$. We can choose Borel measurable function g s.t. $f = g$ $m-$a.e. Apply Fubini's theorem to g and notice that*

$$\int |f| \ dm = \int |g| \ dm \quad \text{and} \quad \int f \ dm = \int g \ dm. \tag{95}$$

12.5 Change of variable theorem

Theorem 12.32 (Change of variable)**.** *Let U, V be open in $\mathbb{R}^n$ and let $\phi : U \to V$ be a C^1 diffeomorphism. Then, for any non-negative Lebesgue measurable function f_n, f on V, we have*

$$\int_V f \ dm = \int_U (f \circ \phi)|J_\phi| \ dm, \tag{96}$$

where

$$J_\phi = \det \phi'. \tag{97}$$

In particular,

$$m(\phi(A)) = \int_A |J_\phi| \ dm, \tag{98}$$

where $A \subset U$ is any Lebesgue measurable set.

13 Normed vector spaces

Definition 13.1 (Norm)**.** *A norm on $(V, \mathbb{F})$ is a function $\|\cdot\| : V \to \mathbb{R}$ with the following properties. For any $v, w \in W, \alpha \in \mathbb{F}$,*

1. *$\|v\| \geq 0$ and $\|v\| = 0$ only for $v = 0$,*
2. *$\|\alpha v\| = |\alpha|\|v\|$,*
3. *$\|v + w\| \leq \|v\| + \|w\|$.*

A vector space with a normed defined on it, $(V, \|\cdot\|)$, is called a normed vector space.

Definition 13.2 (Equivalent norms). *Let $(V, \mathbb{F})$ be a vector space. Two norms on V, $\|\cdot\|_1$, $\|\cdot\|_2$ are said to be equivalent if there exist positive constants C_1, C_2 s.t.*

$$C_1\|v\|_1 \leq \|v\|_2 \leq C_2\|v\|_1 \quad \forall v \in V. \tag{99}$$

Proposition 13.3. *If $(V, \mathbb{R})$ is a finite dimensional vector space of dimension n, then there exists a linear map $i : \mathbb{R}^n \to V$ s.t. if $\|\cdot\|_V$ is any norm on V, then $i : (\mathbb{R}^n, \|\cdot\|) \to (V, \|\cdot\|_V)$ is a homeomorphism where*

$$c_1|x| \leq \|i(x)\|_V \leq c_2|x| \quad \textit{for some positive constants } c_1, c_2. \tag{100}$$

14 Compactness in function spaces

Definition 14.1 (Equicontinuity). *Let (X, d) be a metric space. Let $\mathcal{F}$ be a family of functions $f : X \to \mathbb{R}$. We say $\mathcal{F}$ is equicontinuous on X if given any $\epsilon > 0$ there exists $\delta > 0$ s.t. if $d(p, q) < \delta$, then*

$$|f(p) - f(q)| < \epsilon \quad \forall f \in \mathcal{F}. \tag{101}$$

Theorem 14.2 (Arzela-Ascoli theorem). *Let (X, d) be a compact metric space. Let $K \subset C(X, \mathbb{R})$ be closed, bounded, and equicontinuous, then K is compact.*

Definition 14.3 (Pointwise boundedness). *We say $\mathcal{F} \subset C(X, \mathbb{R})$ is pointwise bounded if given any $p \in X$, there exists M_p s.t.*

$$|f(p)| \leq M_p \quad \forall f \in \mathcal{F}. \tag{102}$$

Corollary 14.4. *Let (X, d) be a compact metric space.*

1. *If $\mathcal{F} \subset C(X, \mathbb{R})$ is bounded and equicontinuous. Then, $\overline{\mathcal{F}}$ is compact in $C(X, \mathbb{R})$.*

2. *If $\mathcal{F} \subset C(X, \mathbb{R})$ is pointwise bounded and continuous, then $\overline{\mathcal{F}}$ is compact in $C(X, \mathbb{R})$.*

15 Density and approximation in function spaces

Proposition 15.1 (Differentiation under the integral sign).

1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $f : [a, b]_t \times X \to \mathbb{C}$ where $-\infty < a < b < \infty$ and $f(t, \cdot)$ is integrable for $t \in [a, b]$. Let*

$$F(t) = \int_X f(t, x)\ d\mu(x). \tag{103}$$

Suppose $\partial_t f(t, x)$ exists $\forall t, x$ and suppose $\exists g \in L^1(X, \mathcal{M}, \mu)$ s.t.

$$|\partial_t f(t, x)| \leq g(x) \quad \forall t, x. \tag{104}$$

Then, F is differentiable and

$$F'(t) = \int_X \partial_t f(t, x)\ d\mu(x). \tag{105}$$

2. *If $\partial_t f(\cdot, x)$ is continuous for each x, then F' is continuous.*

Definition 15.2 (Convolution). *Let $f \in C(\mathbb{R}^n, \mathbb{R})$ and $g \in C_c(\mathbb{R}^n, \mathbb{R})$. Then, the convolution of f and g, $f * g$, is given by*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\ dy = \int_{\mathbb{R}^n} f(y)g(x - y)\ dy. \tag{106}$$

Proposition 15.3. *Let $f \in C(\mathbb{R}^n, \mathbb{R})$. Let $g \in C_c^k(\mathbb{R}^n, \mathbb{R})$ where $k \geq 0$. Then,*

1. $\operatorname{supp} f * g \subset \overline{\operatorname{supp} f + \operatorname{supp} g}$,

2. $f * g \in C^k(\mathbb{R}^n, \mathbb{R})$ *and for $|\alpha| \leq k$, we have*

$$\partial^\alpha (f * g) = f * (\partial^\alpha g), \tag{107}$$

3. *if $f \in C^k(\mathbb{R}^n, \mathbb{R})$, then*

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g). \tag{108}$$

15.1 Approximate identities

Definition 15.4 (Approximate identities)**.** *Take* $g \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ *satisfying the following properties:*

1. $g \geq 0$,
2. $\operatorname{supp} g \subset \overline{B(0,1)}$, *and*
3. $\int_{\mathbb{R}^n} g(x)\ dx = 1$.

We say the sequence of functions (g_k) *is an approximate identity where* g_k *is given by*

$$g_k(x) = k^n g(kx), \quad k = 0, 1, \cdots . \tag{109}$$

Remark 15.5. *Notice that* g_k *satisfy the following properties:*

1. $g_k \geq 0$,
2. $\operatorname{supp} g_k \subset \overline{B(0,1/k)}$, *and*
3. $\int_{\mathbb{R}^n} g_k(x)\ dx = 1$.

Proposition 15.6. *For* $m \geq 0$*, let* $f \in C^m(\mathbb{R}^n, \mathbb{R})$*. Let* (g_k) *be an approximate identity. Define* $f_k \in C^\infty(\mathbb{R}^n, \mathbb{R})$ *by*

$$f_k(x) = (f * g_k)(x). \tag{110}$$

Then, for any compact set $K \subset \mathbb{R}^n$ *and any multiindex* α *with* $|\alpha| \leq m$*, we have* $\partial^\alpha f_k \to \partial^\alpha f$ *uniformly on* K *as* $k \to \infty$*.*

15.2 Approximation theorems

Theorem 15.7 (Weierstrass approximation theorem)**.** *Let* $f \in C([a,b], \mathbb{R})$*. Then, there exists a sequence of polynomials* (p_n) *s.t.* $p_n \to f$ *uniformly on* $[a,b]$ *as* $n \to \infty$*, i.e., the set of all polynomials on* $[a,b]$ *is dense in* $C([a,b], \mathbb{R})$*.*

Definition 15.8 (Algebra)**.** *An algebra of real-valued (resp. complex-valued) functions on a set* X *is a set of functions that is closed under addition, multiplication, and scalar multiplication by* $\alpha \in \mathbb{R}$ *(resp.* $\alpha \in \mathbb{C}$*).*

Definition 15.9 (Self-adjoint). *An algebra of function $f : X \to \mathbb{C}$ where X is a compact metric space is said to be self-adjoint if $f \in \mathcal{A} \implies \overline{f} \in \mathcal{A}$.*

Theorem 15.10 (Stone-Weierstrass theorem (real version)). *Let X be a compact metric space. Let $\mathcal{A} \subset C(X, \mathbb{R})$ be a sub-algebra. Suppose $1 \in \mathcal{A}$ and $\mathcal{A}$ separates points of X, i.e., if $p, q \in X, p \neq q$, then $\exists h_{pq} \in \mathcal{A}$ s.t. $h_{pq}(p) \neq h_{pq}(q)$. Then, $\mathcal{A}$ is dense in $C(X, \mathbb{R})$.*

Theorem 15.11 (Stone-Weierstrass theorem (complex version)). *Let (X, d) be a compact metric space. Let $\mathcal{A} \subset C(X, \mathbb{C})$ be a self-adjoint sub-algebra. Suppose $1 \in \mathcal{A}$ and $\mathcal{A}$ separates points of X, then $\mathcal{A}$ is dense in $C(X, \mathbb{C})$.*

Definition 15.12 (Trigonometric polynomials). *Define the set of all trigonometric polynomials TP to be*

$$TP = \left\{ \sum_{|k| \leq N} a_k e^{ik\theta}, N = 0, 1, \cdots, a_k \in \mathbb{C} \right\}. \tag{111}$$

Proposition 15.13. *Consider the set of periodic functions*

$$C_p([0, 2\pi], \mathbb{C}) = \{f \in C([0, 2\pi], \mathbb{C}), f(0) = f(2\pi)\}.$$

Then the set of all trignometric polynomials TP is dense in $C_p([0, 2\pi], \mathbb{C})$.

16 Existence and uniqueness for systems of ODEs

Consider the following IVP:

$$\frac{dy}{dt} = F(t, y), y(t_0) = y_0. \tag{112}$$

Theorem 16.1 (Local existence). *Let Ω be an open subset of $\mathbb{R}^n$ and let $y_0 \in \Omega$. Let $I \subset \mathbb{R}$ be an open interval containing t_0. Suppose $F : I_t \times \Omega_y \to \mathbb{R}^n$ is continuous and for any compact interval $I_c \subset I$ and compact $K \subset \Omega$ $\exists L > 0$ s.t.*

$$|F(t, y_1) - F(t, y_2)| \leq L|y_1 - y_2| \quad \forall t \in I_c \text{ and } y_1, y_2 \in K. \tag{113}$$

Then, IVP has a C^1 solution on some open interval containing t_0.

Theorem 16.2 (Uniqueness)**.** *Let Ω, I, F as above. Let $I' \subset I$ be an open subinterval containing t_0 on which C^1 solutions y and z of IVP are given. Then $y = z$ on I'.*

Proposition 16.3 (Uniform local existence)**.** *Let Ω, I, F as above. Then, for any fixed compact interval $I_c \subset I$ and fixed compact set $K \subset \Omega$, $\exists T > 0$ s.t. for each $t_0 \in I_c$, $y_0 \in K$, a unique C^1 solution of IVP exists on $[t_0 - T, t_0 + T]$.*

Remark 16.4. *If $F \in C^1(\mathbb{R} \times \mathbb{R}^n)$, then F satisfies uniform local existence when I is any bounded open interval and $\Omega \subset \mathbb{R}^n$ is any convex, bounded, open set.*

Proposition 16.5 (Criterion for global existence)**.** *Let Ω, I, F as above. Suppose that if $J \subset I$ is any bounded open subinterval containing t_0 on which a C^1 solution y exists, there exists a compact set $K \subset \Omega$ s.t. $y(t) \in K \; \forall t \in J$. Then, y extends uniquely to a C^1 solution on all of I.*

Lemma 16.6 (Gronwall's lemma)**.** *Let $I = [a, b]$ and suppose $\alpha, \beta \in C(I, \mathbb{R})$. Assume $u \in C^1(I, \mathbb{R})$ satisfies*

$$u'(t) \leq \alpha(t)u(t) + \beta(t) \; \forall t \in I \quad \text{and} \quad u(a) = u_0. \tag{114}$$

Then,

$$u(t) \leq u_0 \exp\left(\int_a^t \alpha(r)\, dr\right) + \int_a^t \exp\left(\int_s^t \alpha(r)\, dr\right) \beta(s)\, ds \quad \forall t \in I. \tag{115}$$

Proposition 16.7 (Linear energy estimate)**.** *Consider a C^1 solution to the IVP*

$$\frac{dy}{dy} = A(t)y + B(t), y(0) = y_0 \tag{116}$$

on an interval $I \ni 0$. Assume $A \in C(I, M(n, \mathbb{R}))$ and $B \in C(I, \mathbb{R}^n)$. If $||A(t)|| \leq K \; \forall t \in I$, then $\forall t \in I, t \geq 0$, $y(t)$ satisfies

$$|y(t)|^2 \leq e^{(2K+1)t}|y_0|^2 + \int_0^t e^{(2K+1)(t-s)}|B(s)|^2 ds \tag{117}$$

The same formula holds for $t \in I, t \leq 0$, but with $B(s)$ replaced by $B(-s)$ and t replaced by $|t|$ on the right.

Corollary 16.8. *If y_1 and y_2 are C^1 solutions on I, then $y_1 = y_2$.*

17 Introduction to Complex Analysis

17.1 Complex numbers

Definition 17.1. *The field of complex numbers* $\mathbb{C}$ *is a set of ordered pairs* (a, b) *where* $a, b \in \mathbb{R}$ *with operations of addition and multiplication defined by*

$$(a, b) + (c, d) = (a + c, b + d), \tag{118}$$

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc). \tag{119}$$

Remark 17.2. *Define* $i = (0, 1)$*. In addition, if we write* $(a, 0)$ *as* a*, then we have*

$$(a, b) = (a, 0) + (b, 0)(0, 1) = a + ib. \tag{120}$$

Definition 17.3 (Complex conjugate)**.** *The complex conjugate of* z *is given by*

$$\overline{z} = a - ib. \tag{121}$$

Definition 17.4 (Norm on $\mathbb{C}$)**.** *Define*

$$|z| = |(a, b)| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}. \tag{122}$$

17.2 Functions on $\mathbb{C}$

Consider $f : \mathbb{C} \to \mathbb{C}$. We write

$$f(z) = u(z) + iv(z) \tag{123}$$

where $u, v : \mathbb{C} \to \mathbb{R}$.

Definition 17.5 (Analytic function)**.** *Let* $\Omega \subset \mathbb{C}$ *be open. Let* $f : \Omega \to \mathbb{C}$ *be* C^1*, i.e.,* $\operatorname{Re} f, \operatorname{Im} f$ *are* C^1*. We say* f *is analytic (or holomorphic) on* Ω *if*

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \tag{124}$$

exists for all $z \in \Omega$*. If so, we write* $f \in H(\Omega)$*.*

Remark 17.6. *If* $f \in H(\Omega)$*, we write*

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}, \tag{125}$$

and we say f *is complex-differentiable at* z*.*

Theorem 17.7 (Cauchy-Riemann equations)**.** *Let $f : \Omega \to \mathbb{C}$ be C^1. Then $f = u + iv$ is analytic on Ω if and only if f satisfy the Cauchy-Riemann equations on Ω, i.e.,*

$$f_x = \frac{1}{i} f_y, \tag{126}$$

or, equivalently,

$$(u_x, v_x) = (v_y, -u_y), \tag{127}$$

or, equivalently,

$$u_x = v_y \tag{128}$$

$$u_y = -v_x. \tag{129}$$

Remark 17.8. *Let f be as above. Then, we have $f' = f_x = \frac{1}{i} f_y$ on Ω.*

Theorem 17.9 (Cauchy's theorem)**.** *Let Ω be a bounded, connected open set in $\mathbb{C}$ with a C^∞ boundary $\partial\Omega$, oriented positively. If $f \in C^1(\overline{\Omega}, \mathbb{C})$ is analytic in Ω, then*

$$\int_{\partial\Omega} f(z)\ dz = 0. \tag{130}$$

Theorem 17.10 (Cauchy integral formula)**.** *Let f be as above. If $a \in \Omega$, then*

$$f(a) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - a}\ dz. \tag{131}$$

Corollary 17.11 (Smoothness)**.** *Let f, Ω be as above. If $a \in \Omega$, then, for any $n \in \mathbb{N}$,*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{(z - a)^{n+1}}\ dz. \tag{132}$$

Consequently, f is C^∞ in Ω and $f^{(n)}(a) \in H(\Omega)$ $\forall n \in \mathbb{N}$.

Corollary 17.12 (Cauchy's estimate)**.** *Let $f \in H(B(a, R))$ with $R > 0$. Suppose*

$$|f(z)| \leq M \quad \forall z \in B(a, R). \tag{133}$$

Then,

$$|f(z)| \leq \frac{n! M}{R^n}. \tag{134}$$

Theorem 17.13 (Liouville's theorem)**.** *Let $f \in H(\mathbb{C})$, i.e., an entire function, and bounded, i.e., $\exists M > 0$ s.t. $|f(z)| \leq M$ $\forall z \in \mathbb{C}$. Then f is constant.*

Corollary 17.14 (Mean value property I)**.** *Let $f \in C^1(\overline{B(a,r)}, \mathbb{C})$ be analytic in $B(a,r)$. Then*

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\ dt = \frac{1}{2\pi r} \int_{\partial B(a,r)} f(z)\ ds. \tag{135}$$

Here, $ds = |z'(t)|\ dt$.

Corollary 17.15 (Mean value property II)**.** *With the same hypothesis as above, we have*

$$f(a) = \frac{1}{\pi r^2} \int_{B(a,r)} f(z)\ dxdy. \tag{136}$$

Theorem 17.16 (Maximum modulus theorem)**.** *Let Ω be bounded, connected, open in $\mathbb{C}$. Suppose $f \in C(\overline{\Omega}, \mathbb{C})$ and analytic in Ω. Then*

1. *If $a \in \Omega$ and $|f(a)| \geq |f(z)|\ \forall z \in \Omega$, then $|f|$ is constant on Ω.*
2. *As a result, $\sup_{z \in \overline{\Omega}} |f(z)| = \sup_{z \in \partial\Omega} |f(z)|$.*

Theorem 17.17 (Power series expansions)**.** *Let Ω be bounded, connected, open in $\mathbb{C}$ with $\partial\Omega$ oriented positively. Suppose $f \in C^1(\overline{\Omega}, \mathbb{C})$ and f is analytic in Ω. Let $a \in \Omega$ and $r > 0$ s.t. $\overline{B(a,r)} \subset \Omega$. Then, for $z \in B(a,r)$, we have*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \tag{137}$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)}{(w-a)^{n+1}}\ dw. \tag{138}$$

Theorem 17.18 (Morera's theorem)**.** *Let $f : \Omega \to \mathbb{C}$ with Ω open. Assume f is continuous and $\int_T f(z)\ dz = 0$ for every triangular curve T in Ω. Then, f is analytic in Ω.*