MATH 754 Review

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Contents

1 Finite dimensional normed spaces, Hilbert spaces, Banach spaces, Fréchet spaces, topological vector spaces

Definition 1.1 (Normed vector spaces)**.** *A normed vector space* $(V, \|\cdot\|)$ *is a vector space* V *equipped with a function* $\|\cdot\| : V \to \overline{\mathbb{R}^+}$ *s.t.*

1. $\|v\| = 0 \iff v = 0$,
2. *if* $\alpha \in \mathbb{F}$, *then* $\|\alpha v\| = |\alpha|\|V\|$, *and*
3. $\|v + w\| \leq \|v\| + \|w\|$.

Definition 1.2 (Quotient spaces)**.** *Let* V *be a normed vector space and* W *be a closed subspace of* V. *If* $v_1, v_2 \in V$, *we say* $v_1 \sim v_2$ *if* $v_1 - v_2 \in W$. *For* $v \in V$, *we write* $v + W$ *for the equivalence class of* V. *Define*

1. $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$, *and*
2. *if* $\alpha \in \mathbb{C}$, *define* $\alpha(v_1 + W) = \alpha v_1 + W$.

Then, the quotient space

$$V/W = \{v + W : v \in V\} \tag{1}$$

is a vector space.

Definition 1.3 (Norms on quotient spaces)**.** *Define a norm on* V/W *by*

$$\|v + W\| = \inf_{w \in W} \|v + w\| = \inf_{w \in W} \|v - w\| = \operatorname{dist}(v, W). \tag{2}$$

Definition 1.4 (Banach spaces)**.** *A Banach space* $(V, \|\cdot\|)$ *is a normed vector spaces which is complete with respect to the metric* $d : V \times V \to \overline{\mathbb{R}^+}$ *given by* $d(v, w) = \|v - w\|$.

Definition 1.5 (Inner product)**.** *An inner product on a vector space* H *is a map* $(\cdot, \cdot) : H \times H \to \mathbb{C}$ *s.t.*

1. $(u_1 + u_2, v) = (u_1, v) + (u_2, v)$,
2. *if* $\alpha \in \mathbb{C}$, *then* $(\alpha u, v) = \alpha(u, v)$,

3. $(u,v) = \overline{(v,u)}$, *and*

4. $(u,u) \geq 0$ *and* $(u,u) = 0 \iff u = 0$.

Definition 1.6 (Hilbert spaces). *A Hilbert space* $(H, (\cdot,\cdot))$ *is a vector space equipped with an inner product* $(\cdot,\cdot) : H \times H \to \mathbb{C}$ *s.t.* H *is a Banach space with the norm* $\|\cdot\| : H \to \overline{\mathbb{R}^+}$ *given by* $\|u\| = (u,u)^{1/2}$.

Definition 1.7 (Topological vector spaces). *A topological vector space* (V,τ) *is a vector space equipped with a topology* τ *s.t. addition* $+ : V \times V \to V$ *and scalar multiplication* $\cdot : \mathbb{F} \times V \to V$ *are continuous.*

Definition 1.8 (Neighborhood). *A neighborhood of* $a \in V$ *is an open set that contains* a.

Definition 1.9 (Local base). *A collection* $\mathcal{B}$ *of neighborhoods of* 0 *is a local base at* 0 *if every neighborhood of* 0 *contains an element of* $\mathcal{B}$.

Definition 1.10 (Locally convex topological vector spaces). *A locally convex topological vector space* V *is a topological vector space that has a local base at* 0 *consisting of convex sets.*

Definition 1.11 (Fréchet spaces). *A Fréchet space is a locally convex topological vector space whose topology is defined by a complete invariant metric* d, *i.e.,* $d(u+w, v+w) = d(u,v)$.

1.1 Finite dimensional normed vector spaces

Definition 1.12 (Equivalent norms). *Norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *on vector space* V *are equivalent if there exists constants* $a, b > 0$ *s.t.*

$$a\|v\|_1 \leq \|v\|_2 \leq b\|v\|_1. \tag{3}$$

Remark 1.13. *Equivalent norms determine the same open sets.*

Definition 1.14 (Linear isometry). *Let* B_1, B_2 *be normed vector spaces. A map* $i : (B_1, N_1) \to (B_2, N_2)$ *is a linear isometry if* i *is linear, bijective, and* $N_2(i(b)) = N_1(b)$.

Proposition 1.15 (Equivalence of norms). *Let* (X, N) *be a normed vector space s.t.* $\dim X = n \in \mathbb{N}$.

1. *There exists a norm $\tilde{N}$ on $\mathbb{C}^n$ and a linear isometry $i : (\mathbb{C}^n, \tilde{N}) \to (X, N)$.*

2. *Any two norms on (X, N) are equivalent.*

Proposition 1.16. *Let V be a finite dimensional vector space. Then $K \subset V$ is compact if and only if K is closed and bounded.*

Corollary 1.17. *The norm closed unit ball $\{v \in V : \|v\| \leq 1\}$ in any finite dimensional normed vector space is compact.*

Theorem 1.18. *Let V be an infinite dimensional normed-vector space. Then, the norm closed unit ball $\{v \in V : \|v\| \leq 1\}$ is not compact.*

1.2 Hilbert spaces

Proposition 1.19 (Cauchy-Schwarz)**.** *Let $(H, (\cdot, \cdot))$ be a Hilbert space. Then,*

$$|(u, v)| \leq \|u\|\|v\|. \tag{4}$$

Proposition 1.20 (Parallelogram law)**.** *Let H be a Hilbert space. Then,*

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2. \tag{5}$$

1.2.1 Orthogonality

Definition 1.21. *If $u, v \in H$, we say u is orthogonal to v and write $u \perp v$ if $(u, v) = 0$.*

Definition 1.22 (Convex sets)**.** *Let V be a vector space. A set $K \subset V$ is convex if*

$$tx + (1 - t)y \in K \quad \forall x, y \in K, t \in [0, 1]. \tag{6}$$

Proposition 1.23. *Let H be a Hilbert space, let $K \subset H$ be closed and convex and let $x \in H$. Then, there exists a unique $z \in K$ s.t.*

$$\|x - z\| = \operatorname{dist}(x, K) = \inf_{y \in K} \|x - y\| = d. \tag{7}$$

Notation 1.24.

1. *Let $x \in H$. Let $K \subset H$ be a closed subspace. Let $P_K x$ be the point in K that is closest to x. We can write*

$$x = P_K x + (I - P_K)x. \tag{8}$$

2. *Write $K^\perp = \{u \in H : (u, v) = 0 \ \forall v \in K\}$.*

Proposition 1.25. *$x - P_K x \in K^\perp$.*

Proposition 1.26 (Orthogonal decomposition of H)**.** *Let $K \subset H$ be a closed subspace, then any $x \in H$ can be written uniquely as*

$$x = x_1 + x_2 \quad \text{where } x_1 \in K, x_2 \in K^\perp. \tag{9}$$

In fact, we have

$$x_1 = P_K x, x_2 = (I - P_K)x. \tag{10}$$

We write $H = K \oplus K^\perp$.

Definition 1.27 (Projections)**.** *Let X be a vector space. A linear map $E : X \to X$ with $E^2 = E$ is called a projection.*

Proposition 1.28. *Let $E : X \to X$ be a projection. Then, we can write $X = M \oplus N$ where*

$$M = \{x \in X : Ex = x\}, N = \{x \in X : Ex = 0\}. \tag{11}$$

Proposition 1.29. *$P_{K^\perp} = I - P_K$.*

Proposition 1.30. *Both P_K and $P_{K^\perp}$ are linear (and are thus projections).*

Definition 1.31 (Orthogonal projection)**.** *We say $P_K : H \to K$ is the orthogonal projection of H onto K.*

1.2.2 Riesz Representation theorem

Theorem 1.32. *Riesz Representation theorem Let $\varphi : H \to \mathbb{C}$ be a continuous, linear functional. Then, there exists a unique $f \in H$ s.t.*

$$\varphi(u) = (u, f) \quad \forall u \in H. \tag{12}$$

Definition 1.33 (Dual space of H)**.** *The dual space of H is*

$$H' = \{\varphi : H \to \mathbb{C}, \varphi \text{ is continuous, linear}\}. \tag{13}$$

Proposition 1.34. *Riesz Representation theorem gives a map $T : H' \to H$, i.e., $\varphi \mapsto f$ where T is a antilinear isometry of H' onto H.*

1.2.3 Orthornormal sets in Hilbert spaces

Definition 1.35. *A set $\{e_\alpha : \alpha \in A\} \subset H$ is an orthonormal set if*

1. $|e_\alpha| = 1\ \forall \alpha \in A$, *and*
2. $(e_\alpha, e_\beta) = 0$ *if* $\alpha \neq \beta$.

Theorem 1.36 (Bessel's inequality)**.** *Let $\{e_\alpha : \alpha \in A\}$ be any orthonormal set in H. Let $x \in H$. Then,*

$$\sum_{\alpha \in A} |(x, e_\alpha)|^2 \leq \|x\|^2 \tag{14}$$

and

$$\sum_{\alpha \in A} (x, e_\alpha) e_\alpha \tag{15}$$

converges in H.

Definition 1.37 (Maximal orthonormal sets)**.** *An orthonormal set $S \subset H$ is maximal if it is not contained in any larger orthonormal set.*

Proposition 1.38. *Let H be a Hilbert space. Any orthonormal set $\mathcal{O}$ can be extended to a maximal orthonormal set.*

Definition 1.39. *A Banach space B is separable if it has a countable dense subset.*

Proposition 1.40 (Maximal orthonormal sets in Hilbert spaces)**.** *Let H be a separable Hilbert space. Then, we can construct a countable maximal orthonormal set.*

1.2.4 $\ell^2(A)$

Definition 1.41. *Let A be any set. Define*

$$\ell^2(A) = \left\{ \varphi : A \to \mathbb{C} : \sum_{\alpha \in A} |\varphi(\alpha)|^2 < \infty \right\}, \tag{16}$$

where the summation is interpreted as taking supremum over all finite sums.

Theorem 1.42 (Riesz-Fisher theorem)**.** *Let $S = \{e_\alpha : \alpha \in A\} \subset H$ be an orthonormal set. If $\varphi \in \ell^2(A)$, then there exists $x \in H$ s.t.*

$$(x, e_\alpha) = \varphi(\alpha) \quad \forall \alpha \in A. \tag{17}$$

Definition 1.43 (Unitary equivalence)**.** *Let H_1, H_2 be Hilbert spaces. A linear map $U : H_1 \to H_2$ is a unitary equivalence if U is bijective and preserves inner product, i.e.,*

$$(x, y)_{H_1} = (Ux, Uy)_{H_2}. \tag{18}$$

Theorem 1.44. *Let $S = \{e_\alpha : \alpha \in A\} \subset H$ be a maximal orthonormal set. Then $U : H \to \ell^2(A)$ given by $x \mapsto \varphi$ where*

$$\varphi(\alpha) = (x, e_\alpha) \quad \forall \alpha \in A \tag{19}$$

is a unitary equivalence.

Theorem 1.45 (Parseval's formula)**.** *Let $S = \{e_\alpha : \alpha \in A\} \subset H$ be a maximal orthonormal set. If $x, y \in H$, then*

$$(x, y) = \sum_{\alpha \in A} (x, e_\alpha)\overline{(y, e_\alpha)}. \tag{20}$$

1.3 Banach spaces

Proposition 1.46. *Let V be a Banach space and let $W \subset V$ be a closed subspace. Then, the quotient space V/W is complete.*

Proposition 1.47. *Let $T : V \to W$ be linear. Then, T is continuous if and only if there exists $C > 0$ s.t.*

$$\|Tv\| \le C\|v\| \quad \forall v \in V. \tag{21}$$

1.4 Topological vector spaces and Fréchet spaces

Proposition 1.48. *Let V be a topological vector space, with topologies τ_1, τ_2 and local bases at 0 given by $\mathcal{B}_1, \mathcal{B}_2$, respectively. Then $\tau_1 = \tau_2$ if and only if every element of $\mathcal{B}_1$ contains an element of $\mathcal{B}_2$ and vice versa.*

1.4.1 Seminorms

Definition 1.49 (Seminorms). *A seminorm on a vector space V is a function $p : V \to \overline{\mathbb{R}^+}$ s.t.*

1. $p(\alpha v) = |\alpha| p(v)\ \forall \alpha \in \mathbb{C}\ \forall v \in V$, *and*
2. $p(v + w) \leq p(v) + p(w)\ \forall v, w \in V$.

Definition 1.50 (Separating family of seminorms). *A family $\mathcal{F}$ of seminorms on V is called separating if given any $v \in V$ with $v \neq 0$ there exists $p \in \mathcal{F}$ s.t. $p(v) \neq 0$.*

Theorem 1.51 (Description of the topology using seminorms). *Let $\mathcal{P}$ be a separating family of seminorms on a vector space V. For $p \in \mathcal{P}$ and $n \in \mathbb{N}$, let*

$$V(p, n) = \left\{ x \in V : p(x) < \frac{1}{n} \right\}. \tag{22}$$

Let $\mathcal{B}$ be the set of all finite intersections of the $V(p_n)$. Let τ be the set of all unions of translates of elements of $\mathcal{B}$. Then, (V, τ) is a locally convex topological vector space. Furthermore, $p \in \mathcal{P}$ is continuous and $\mathcal{B}$ is a local base at 0.

Theorem 1.52 (Description of the topology using a metric). *Let $\mathcal{P}$ be a countable, separating family of seminorms on a vector space V. Define an invariant metric $d : V \times V \to \overline{\mathbb{R}^+}$ by*

$$d(u, v) = \sum_{j=1}^{\infty} \frac{2^{-j} p_j(u - v)}{1 + p_j(u - v)}. \tag{23}$$

Proposition 1.53. *Let $\mathcal{P}$ be a countable, separating family of seminorms on a vector space V. Let τ_s be the topology determined by the seminorms and let τ_m be the topology determined by the metric. Then, $\tau_s = \tau_m$.*

Proposition 1.54. *Let (V, τ) be a locally convex topological vector space with τ determined by a separating family of seminorms $\mathcal{P} = \{p_\alpha : \alpha \in A\}$. Then,*

1. *a sequence $v_n \to v$ if and only if it converges in every seminorm, i.e., $p_\alpha(v_n - v) \to 0\ forall \alpha \in A$, and*

2. (v_n) *is Cauchy if and only if it is Cauchy in every seminorm, i.e., given* $\varepsilon > 0$ *and* $\alpha \in A$, *there exists* $N \in \mathbb{N}$ *s.t. if* $m, n \geq N$, *then* $p_\alpha(v_m - v_n) < \varepsilon$.

Proposition 1.55. *Let* V *be a Fréchet space with topology given by* $\{p_j\}_{j=1}^{\infty}$. *Let* $\omega : V \to \mathbb{C}$ *be linear. Then,* ω *is continuous if and only if there exists* C, N *s.t.*

$$|\omega(v)| \leq C \sum_{j=1}^{N} p_j(v) \quad \forall v \in V. \tag{24}$$

2 Duality

2.1 Hahn-Banach theorem and its corollaries

Theorem 2.1 (Hahn-Banach theorem)**.** *Let* X *be a vector space over* $\mathbb{F}$. *Let* p *be a seminorm. Let* $M \subset X$ *be a subspace. Let* $f : M \to \mathbb{F}$ *be linear s.t.*

$$|f(x)| \leq p(x) \ \forall x \in M. \tag{25}$$

Then, there exists an extension $\tilde{f} : X \to \mathbb{F}$ *linear s.t.*

$$|\tilde{f}(x)| \leq p(x) \quad \forall x \in X. \tag{26}$$

Corollary 2.2. *Let* X *be a normed vector space over* $\mathbb{C}$. *Let* $x_0 \in X$. *Then, there exists* $\Lambda \in X'$ *s.t.*

$$\Lambda x_0 = |x_0| \quad \text{and} \quad |\Lambda x| \leq |x| \quad \forall x \in X. \tag{27}$$

Remark 2.3. *So, if* $X \neq \{0\}$ *is a normed vector space, then*

1. $X' \neq \{0\}$ *and*

2. *if* $x \in X$ *and* $f(x) = 0 \ \forall f \in X'$, *then* $x = 0$.

Corollary 2.4. *Let* X *be a normed vector space. Let* $M \subset X$ *be a subspace. Let* $x_0 \in X$. *Then,* $x_0 \in \overline{M}$ *if and only if given* $f \in X'$ *and* $f = 0$ *on* M, *we have* $f(x_0) = 0$.

Corollary 2.5. *Let* X *be a normed vector space. Let* $M \subset X$ *be a subspace. Then,* M *is dense in* X *if and only if given* $f \in X'$ *and* $f = 0$ *on* M, *we have* $f = 0$ *on* X.

Corollary 2.6. *Let* X *be a normed vector space. If* X' *is separable, then* X *is separable.*

2.2 Reflexive Banach spaces

Definition 2.7 (J)**.** *Let V be a Banach space. There is a natural map $J : V \to V''$ given by*

$$Jv(\omega) = \omega(v) \in \mathbb{C} \quad \forall \omega \in V'. \tag{28}$$

Proposition 2.8. *The map $J : V \to J(V) \subset V''$ is an isometry onto $J(V)$.*

Definition 2.9 (Reflexive Banach spaces)**.** *A Banach space V is reflexive if $J : V \to V''$ is surjective.*

2.3 Weak topologies

Definition 2.10 (Subbase)**.** *Let (X, τ) be a topological space. Say $\mathcal{B}_s \subset \tau$ is a subbase for τ if the set $\mathcal{B}$ of all finite intersections of the elements of $\mathcal{B}_s$ is a base for τ, i.e., every element of τ is a union of elements of $\mathcal{B}$.*

Theorem 2.11 (Tychonov's theorem)**.** *If x_α, $\alpha \in A$ are compact Hausdorff, then $\prod_{\alpha \in A} X_\alpha$ is compact Hausdorff with respect to the product topology.*

Definition 2.12 (Weak topologies on X)**.** *Let X be a set. Let $\mathcal{F}$ be a family of maps $f : X \to Y_f$ where Y_f is a topological space. Define a topology τ_w on X to have subbase*

$$\mathcal{B}_s = \{f^{-1}(V) : V \subset Y_f \text{ open, } f \in \mathcal{F}\}. \tag{29}$$

Call τ_w the weak topology induced by $\mathcal{F}$ or the $\mathcal{F}-$topology.

Remark 2.13. *A local subbase at $0 \in X$ for τ_w is given by the sets*

$$\{x \in X : p_\omega(x) < \varepsilon, \omega \in X', \varepsilon > 0\} \quad \text{where } p_\omega(x) = |\omega(x)|. \tag{30}$$

So, the weak topology τ_w on X is a seminorm topology.

Corollary 2.14. *$x_n \rightharpoonup 0$ weakly in X if and only if given any $\omega \in X'$, we have $\omega(x) \to 0$.*

Remark 2.15. *If $x_n \to 0$ strongly, then $x_n \rightharpoonup 0$ weakly in X.*

2.4 Weak* topologies

Definition 2.16 (Weak* topologies on X)**.** *Let X be a topological vector space with dual X'. Let $\mathcal{F}$ be the family*

$$\mathcal{F} = \{J_x : X' \to \mathbb{C}, x \in X\} \quad \text{where } J_x(\omega) = \omega(x) \quad \forall \omega \in X'. \tag{31}$$

The weak topology on X', τ^*, is the $\mathcal{F}-$topology on X'.*

Remark 2.17. *A local subbase at $0 \in X'$ for τ^* is given by the sets*

$$(J_x)^{-1}(B(0,\varepsilon)) = \{\omega \in X' : |J_x(\omega)| = |\omega(x)| < \varepsilon\}. \tag{32}$$

So, the weak topology τ^* is a seminorm topology.*

Corollary 2.18. *$\omega_n \to 0$ in τ^* if and only if given any $x \in X$, we have $\omega_n(x) \to 0$.*

2.5 Compactness

2.5.1 Weak* compactness

Theorem 2.19 (Banach-Alaoglu)**.** *Let X be a Banach space. The norm closed unit ball B in X',*

$$B = \{\Lambda \in X' : |\Lambda x| \leq 1 \ \forall |x| \leq 1\} \tag{33}$$

is compact in the weak topology.*

2.5.2 Weak* sequential compactness

Corollary 2.20. *Let X be a separable Banach space. The norm closed unit ball B in X' is sequentially compact in the weak* topology.*

2.5.3 Weak compactness

Proposition 2.21. *If X is a reflexive Banach space, then $J : X \to X''$ is a homeomorphism of (X, τ) onto (X'', τ^*) where τ is the weak topology on X and τ^* is the weak* topology on X''.*

Corollary 2.22. *Let X be a reflexive Banach space. The norm closed unit ball B in X is compact in the weak topology.*

2.5.4 Weak sequential compactness

Corollary 2.23. *Let X be a separable, reflexive Banach space. The norm closed unit ball B in X is sequentially compact.*