Bloch Equations

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August 28, 2024

1 Introduction

For the rest of this note, we follow Bloch's notation and denote by $\mathbf{M}$ the nuclear polarization, i.e, the net nuclear magnetic moment per unit volume.

2 Larmor precession

We assume the reader has basic familiarity with electromagnetism and classical mechanics. We start from the classical equation

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}, \tag{1}$$

where $\mathbf{L}$ denote the angular momentum of all the nuclei of interest and $\boldsymbol{\tau}$ denote the net torque acting upon all the nuclei. Next, we recall another classical equation relating the aligning torque $\boldsymbol{\tau}$ acting on a dipole from an externally applied magnetic field $\mathbf{B}$, which gives us

$$\boldsymbol{\tau} = \mathbf{M} \times \mathbf{B}. \tag{2}$$

Lastly, recall the gyromagnetic ratio γ, which gives us

$$\mathbf{M} = \gamma \mathbf{L}. \tag{3}$$

Hence, from (1), (2), and (3), we have

$$\frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B}. \tag{4}$$

3 Relaxation

The equation in (4) describe the Larmor precession of **M** in an external magnetic field **B**. However, Bloch noticed that the NMR nuclear induction signal decayed quickly. This motivated Bloch to propose the concept of relaxation, which describes how **M** returns to its initial alignment parallel to **B** with some equilibrium magnitude M_0.

Relaxation is characterized by two constants, i.e., T_1, the longitudinal relaxation time, and T_2, the transversal relaxation time. Bloch assumed that both T_1 and T_2 relaxations follow first-order kinetics, i.e.,

$$\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}, \tag{5}$$

$$\frac{dM_x}{dt} = -\frac{M_x}{T_2}, \tag{6}$$

$$\frac{dM_y}{dt} = -\frac{M_y}{T_2}. \tag{7}$$

4 Bloch Equations

Since (4) describes the rate of change in **M** solely due to Larmor precession and (5), (6), and (7) describe the rate of change in **M** solely due to relaxation, we can add (4), (5), (6), and (7), and have

$$\frac{dM_z}{dt} = \gamma(M_x B_y - M_y B_x) - \frac{M_z - M_0}{T_1}, \tag{8}$$

$$\frac{dM_x}{dt} = \gamma(M_y B_z - M_z B_y) - \frac{M_x}{T_2}, \tag{9}$$

$$\frac{dM_y}{dt} = \gamma(M_z B_x - M_x B_z) - \frac{M_y}{T_2}. \tag{10}$$

5 Solutions

For the sake of simplicity, consider the case of a constant external magnetic field

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix}. \tag{11}$$

Notice that under the assumption in (11), (8) can be written as

$$\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}, \tag{12}$$

whose solution is given by

$$M_z(t) = M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1}). \tag{13}$$

Notice that as $t \to 0$, we have $M_z(t) \to M_0$.

To simplify (9) and (10), use assumption (11) and recall Larmor equation

$$\omega_0 = \gamma B_0, \tag{14}$$

and we have

$$\frac{dM_x}{dt} = \gamma M_y B_z - \frac{M_x}{T_2} = \omega_0 M_y - \frac{M_x}{T_2}, \tag{15}$$

$$\frac{dM_y}{dt} = -\gamma M_x B_z - \frac{M_y}{T_2} = -\omega_0 M_x - \frac{M_y}{T_2}, \tag{16}$$

whose solution is given by

$$M_x(t) = e^{-t/T_2}(M_x(0)\cos\omega t + M_y(0)\sin\omega t), \tag{17}$$

$$M_y(t) = e^{-t/T_2}(M_y(0)\cos\omega t - M_x(0)\sin\omega t). \tag{18}$$