# Deformable Image Registration

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Realistically, images are defined on discretely sampled grids. However, for the sake of simplicity, we assume each image is defined on a continuously sampled coordinate system instead.

**Definition 1** (Image domain). Let $\Omega \subseteq \mathbb{R}^3$ be the image domain on which both fixed and moving images are defined.

**Definition 2** (Fixed and moving images). Let

$$f : \Omega_f \to \mathbb{R}, m : \Omega_m \to \mathbb{R}$$

denote fixed and moving images, respectively, where the value at each voxel is a scalar, i.e., a grayscale intensity.

**Definition 3** (Deformation field). Let

$$\phi : \Omega_f \to \Omega_m$$

be a deformation field that warps the moving image to the fixed image.

**Remark 4** (Matrix definition of deformation fields). Typically, in practice, $\phi$ is represented by a 3D matrix of shape $H \times W \times D$. Let $\phi_{hwd}$ denote the entry of the matrix at $(h, w, d)$. In this case, $(h, w, d) \in \Omega_f$ is the voxel location in the fixed image domain whereas $\phi_{hwd} \in \Omega_m$ is the corresponding voxel location in the moving image domain.

**Remark 5.** Definition 3 might look strange since $\phi$ is defined on $\Omega_f$ and takes values in $\Omega_m$ but it represents the warping of the moving image to the fixed image. There are two ways to explain this.

First, consider applying $\phi$ to $m$, i.e., $m \circ \phi$. Since $\phi : \Omega_f \to \Omega_m$ and $m : \Omega_m \to \mathbb{R}$. We know

$$m \circ \phi : \Omega_f \to \mathbb{R}$$

is an image defined in the image domain of the fixed image.

In addition, we can also consider how deformation is applied to the moving image in practice. For each voxel location $\mathbf{p} \in \Omega_f$, $\phi(\mathbf{p}) \in \Omega_m$ represents the corresponding voxel location in the image domain of the moving image. $\phi(\mathbf{p})$ is also the voxel location from which we will sample an intensity $m(\phi(\mathbf{p})) \in \mathbb{R}$ for the voxel $\mathbf{p} \in \Omega_f$.

**Definition 6** (Displacement field). We define displacement field as a collection of vectors $u(\mathbf{p})$ at each voxel location $\mathbf{p} \in \Omega_f$ that needs to be added to $\mathbf{p}$ to obtained the corresponding voxel location $\phi(\mathbf{p}) \in \Omega_m$.

**Remark 7** (Matrix definition of displacement fields). Similar to Remark 4, we can write $u$ in terms of a 3D matrix.

**Proposition 8.** Using the matrix definition in Remark 4 and 7, we have

$$\phi = Id + u$$

where $Id : \mathbb{R}^3 \to \mathbb{R}^3$ is the identity transformation and $u : \Omega_f \to \mathbb{R}^3$.

**Remark 9.** (Warping point clouds). Let $\{\mathbf{p}_i\}_{i=1}^N$ denote a collection of $N$ points defined on $\Omega_f$. Suppose $u : \Omega_f \to \mathbb{R}^3$ as in Proposition 8. Then, we can warp each $\mathbf{p}_i \in \Omega_f$ to the corresponding voxel location $\tilde{\mathbf{p}}_i \in \Omega_m$, i.e.,

$$\tilde{\mathbf{p}}_i = \mathbf{p}_i + u(\mathbf{p}_i).$$