All Hilbert spaces are ℓ^2-spaces

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1 Introduction

I would like to start with a little update about myself. I started taking MATH 754: Introductory Functional Analysis this semester. So far, I have enjoyed it. There is one thing in particular that really surprises me, i.e., that all Hilbert spaces are ℓ^2-spaces. Before we get started, here are some definitions and propositions.

2 ℓ^2-spaces

In this section, we define ℓ^2-space and discuss its relation to L^2-spaces, the square-integrable functions.

Definition 2.1 (ℓ^2-spaces)**.** *Let A be any set (possibly uncountable). Define*

$$\ell^2(A) = \{\phi : A \to \mathbb{C} : \sum_{\alpha \in A} |\phi(\alpha)|^2 < \infty\}, \tag{1}$$

where the infinite sum is interpreted as the supremum over all finite sums.

Remark 2.2. *Fix $\phi \in \ell^2(A)$. Let $\sum_{\alpha\in A} |\phi(\alpha)|^2 = L < \infty$. Let*

$$B = \{\alpha \in A : \phi(\alpha) \neq 0\} = \bigcup_{n=1}^{\infty} E_n, \text{ where } E_n = \left\{\alpha \in A : \phi(\alpha) \geq \frac{1}{n}\right\}. \tag{2}$$

Suppose $\alpha_1, \cdots, \alpha_k \in E_n$, we then have

$$k \leq Ln^2. \tag{3}$$

Therefore, E_n is finite and B is countable.

Proposition 2.3. *Using the fact that* B *is countable, we can verify that*

$$\langle \phi, \psi \rangle = \sum_{\alpha \in A} \phi(\alpha)\overline{\psi(\alpha)} \quad \forall \phi, \psi \in \ell^2(A) \tag{4}$$

defines an inner product on $\ell^2(A)$.

Definition 2.4 (Norm on ℓ^2-spaces)**.** *Define* $|\cdot|_{\ell^2} : \ell^2(A) \to \overline{\mathbb{R}^+}$ *to be the norm induced by the inner product as in Proposition 2.3, i.e.,*

$$|\phi|_{\ell^2} = \left(\sum_{\alpha \in A} |\phi(\alpha)|^2 \right)^{\frac{1}{2}} \quad \forall \phi \in \ell^2(A). \tag{5}$$

Proposition 2.5. *The vector space* $\ell^2(A)$ *equipped with the inner product as in Proposition 2.3 is a Hilbert space.*

Remark 2.6. *Notice that*

$$\ell^2(A) = L^2(A, \mathcal{P}(A), \mu_A) \tag{6}$$

as sets, where $\mathcal{P}(A)$ *is the power set of* A *and* μ_A *is the counting measure.*

3 More prerequisites

In this section, we list a couple of definitions and some theorems that are rather well-known in functional analysis.

Theorem 3.1 (Riesz-Fisher theorem)**.** *Let* H *be a Hilbert space. Let*

$$\{e_\alpha : \alpha \in A\} \subset H \tag{7}$$

be an orthonormal set (possibly uncountable). If $\phi \in \ell^2(A)$, *then* $\exists x \in H$ *s.t.*

$$\langle x, e_\alpha \rangle = \phi(\alpha) \quad \forall \alpha \in A. \tag{8}$$

Definition 3.2. *Let* H *be a Hilbert space. An orthonormal set is maximal if it is not contained in any larger orthonormal set.*

Proposition 3.3. *Let* H *be a Hilbert space. Let* $S = \{e_\alpha : \alpha \in A\} \subset H$ *be an orthonormal set. Then,* S *is maximal if and only if* $\overline{\operatorname{span} S} = H$, *in which case, if* $x \in H$,

$$x = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha \tag{9}$$

Definition 3.4 (Unitary equivalence)**.** *Let H_1, H_2 be inner product spaces. A linear map $U : H_1 \to H_2$ is a unitary equivalence if U is bijective and preserves inner product, i.e.,*

$$\langle x, y\rangle_{H_1} = \langle Ux, Uy\rangle_{H_2}. \tag{10}$$

Theorem 3.5. *An orthonormal set $\{e_\alpha : \alpha \in A\}$ is maximal if and only if $U : H \to \ell^2(A)$ given by $Ux = \phi$ where*

$$\phi(\alpha) = \langle x, e_\alpha\rangle \quad \forall \alpha \in A \tag{11}$$

is a linear isometry of H onto $\ell^2(A)$.

Proof. Notice that U is linear by construction due to the linearity of inner product in the first argument. Furthermore, U is also onto by construction due to Riesz-Fisher theorem. To finish, we only need to show

$$\|x\|^2 = \sum_{\alpha \in A} |\langle x, e_\alpha\rangle|^2 \iff S \text{ maximal}. \tag{12}$$

1. ($\Longrightarrow$). If S is not maximal, then $\exists x \neq 0$ s.t.

$$\langle x, e_\alpha\rangle = 0 \quad \forall \alpha \in A. \tag{13}$$

Then, $\|x\|^2 \neq 0$ but $\sum_{\alpha \in A} |\langle x, e_\alpha\rangle|^2 = 0$.

2. ($\Longleftarrow$). By Proposition 3.3, we can write

$$x = \sum_{\alpha \in A} \langle x, e_\alpha\rangle e_\alpha. \tag{14}$$

Recall that by Remark 2.2, the infinite sum is in fact a countable sum. Hence, by orthogonality and the continuity of inner product in the first argument, we have

$$\langle x, x\rangle = \left\langle \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i, \sum_{j=1}^{\infty} \langle x, e_j\rangle e_j \right\rangle = \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2. \tag{15}$$

□

Proposition 3.6. *Let H_1, H_2 be inner product spaces. Then, $U : H_1 \to H_2$ is a bijective linear isometry if and only if U is a unitary equivalence.*

We now state the desired result, which makes explicit the exact sense in which all Hilbert spaces are ℓ^2-spaces.

Theorem 3.7. *Let H be a Hilbert space. If $\{e_\alpha : \alpha \in A\} \subset H$ is a maximal orthonormal set, then $U : H \to \ell^2(A)$ given by $Ux = \phi$ where*

$$\phi(\alpha) = \langle x, e_\alpha \rangle \quad \forall \alpha \in A \tag{16}$$

is a unitary equivalence.

Proof. By Theorem 3.5, we see that U is a bijective linear isometry. By Proposition 3.6, we see that U is a unitary equivalence. □